arXiv:cond-mat/9812286v2 [cond-mat.stat-mech] 13 Jun 1999

Supersymmetric Field Theory of Non–Equilibrium Thermodynamic System

Alexander I. Olemskoi and Valerii A. Brazhnyi [1]

Physical Electronics Department, Sumy State University
2, Rimskii-Korsakov Str., 244007 Sumy UKRAINE

Abstract

On the basis of Langevin equation the optimal SUSY field scheme is formulated to discribe a non–equilibrium thermodynamic system with quenched disorder and non–ergodicity effects. Thermodynamic and isothermal susceptibilities, memory parameter and irreversible response are determined at different temperatures and quenched disorder intensities.

PACS: 05.70.Ln; 11.30.Pb; 64.60.–i

Keywords: Order Parameter; Fluctuatuon; Quenched disorder; Non–ergodicity Parameter

1 Introduction

Recently, the microscopic theory of non–equilibrium thermodynamic systems with broken ergodicity and exhibiting the memory effect, has been the subject of major interest. Spin glasses [1] and random heteropolymers [2], that have received much consideration, are well-known examples of such systems. Despite the bulk of theoretical studies had been employed the replica method to approach the problem analytically, there is increasing interest in alternative methods that go beyond the replica trick. The supersymmetry (SUSY) approach, evolved within stochastic dynamics theory governed by Langevin equation, gives a good example of method of such kind.

According to this method, generating functional of Langevin dynamical system is represented as a functional integral over the superfields with Euclidean action by means of introducing Grassmann anticommutating variables. These variables and their products

[1] olemskoi@ssu.sumy.ua, valera@ssu.sumy.ua

serve as a basis for superfields with components that involve Grassmann fields along with real(complex)–valued ones.

As it was shown in [3], static replica treatment of spin systems bears striking similarity with the dynamics expressed in terms of superspace within the framework of the SUSY method. The latter is based on using of nilpotent variables. Two–point correlator of SUSY field can be written in the form of expansion with coefficients that give correlators of observables such as structure factor S, and retarded and advanced Green functions $G_{\mp}$. The memory and nonergodicity effects are allowed for by incorporating the additional terms q, Δ into the correlators. The resulting self–consistent SUSY scheme gives a set of equations for memory q and nonergodicity Δ parameters to be determined as functions of temperature T and quenched disorder h.

For SUSY scheme formulation SUSY as a gauge field needs to be reduced to irreducible components. By analogy with electromagnetic field, that can be splitted into vector and scalar fields, 4–component SUSY field can be divided into chiral components that consist of regular and Grassmann constituents [4]. In Sections 2 and 3 SUSY field will be reduced to 2–component nilpotent field. The latter has an advantage over conventional SUSY representation because its components have an explicit physical meaning of order parameter and conjugate field (or amplitude of its fluctuation). This rises the question as to optimal choice of the basis for making expansion of SUSY correlators. Currently, two types of such basis are known [5,6]. The first one contains 3 components: advanced and retarded Green functions $G_{\pm}$ and structure factor S. The second basis corresponds to the proper 4–component SUSY field and contains 5 components which, in addition to the above mentioned ones, include a couple of mutually conjugated correlators of the Grassmann fields. In Sect.4 it will be shown that the second basis can be reduced to the first one.

The work is organized as follows. In Sect.2, the simplest field scheme is formulated in terms of the 2–component nilpotent fields with second component is taken to be either an amplitude of fluctuation or a conjugate force (see subsections 2.1 and 2.2). In Sect.3 the above–discussed method for reduction of the 4–component proper SUSY field to different 2–component forms is presented. In Sect.4 we show that the reduction results in decrease of the number of components of the SUSY correlator basis, due to the fact that the conjugate correlators of the Grassmann fields are equal to the retarded Green function in accordance with Ward identities. The SUSY perturbation theory, stated for both cubic and quadratic anharmonicities in Sect.5, makes expressions for the SUSY self–energy function simple to calculate. This function enters the SUSY Dyson equation derived in Sect.6 on the basis of effective Lagrangians for both thermodynamic systems with quenched disorder and random heteropolymers. Nonergodicity and memory effects are investigated in Sect.7. The corresponding self–consistent equations are obtained. Behavior of a non–equilibrium thermodynamic system for various values of temperature and quenched disorder intensity is analyzed in Sect.8. Appendices A, B, C provide details concerning the SUSY formalism

under consideration.

2 Two–component SUSY representation

Let us start with the simplest stochastic Langevin equation [7] governing the spatiotemporal evolution of order parameter $\eta(\mathbf{r},t)$:

$$\dot{\eta}(\mathbf{r},t) - D\nabla^2\eta = -\gamma(\partial V/\partial\eta) + \zeta(\mathbf{r},t), \tag{1}$$

where the dot stands for derivative with respect to time, $\nabla \equiv \partial/\partial\mathbf{r}$, D is the diffusion–like coefficient, γ is the kinetic coefficient, $V(\eta)$ is the synergetic potential (Landau free energy), $\zeta(\mathbf{r},t)$ is a Gaussian stochastic function subjected to the white noise conditions

$$\langle\zeta(\mathbf{r},t)\rangle_0 = 0, \qquad \langle\zeta(\mathbf{r},t)\zeta(\mathbf{0},0)\rangle_0 = \gamma T\delta(\mathbf{r})\delta(t), \tag{2}$$

where the angular brackets with subscript 0 denote averaging over the Gaussian probability distribution of ζ, T is the intensity of the noise (the temperature of thermostat).

Further, it is convenient to measure time t, coordinate $\mathbf{r}$, synergetic potential V, and stochastic variable ζ, in units $t_s \equiv (\gamma T)^2/D^3$, $r_s \equiv \gamma T/D$, $V_s \equiv D^3/\gamma^3T^2$, $\zeta_s \equiv D^3/(\gamma T)^2$ respectively. The equation of motion (1) then reads

$$\dot{\eta}(\mathbf{r},t) = -\delta V/\delta\eta + \zeta(\mathbf{r},t), \tag{3}$$

where short notation is used for the variational derivative

$$\delta V/\delta\eta \equiv \delta V\{\eta\}/\delta\eta = \partial V(\eta)/\partial\eta - \nabla^2\eta, \qquad V\{\eta\} \equiv \int\left[V(\eta) + \frac{1}{2}\left(\nabla\eta\right)^2\right]\mathrm{d}\mathbf{r}, \tag{4}$$

the coefficient γT in Eq.(2) becomes unity and the distribution of variable ζ takes the Gaussian form

$$P_0\{\zeta\} \propto \exp\left(-\frac{1}{2}\int\zeta^2(\mathbf{r},t)\mathrm{d}\mathbf{r}\mathrm{d}t\right). \tag{5}$$

The basis for construction of the field scheme is the generating functional [8]

$$Z\{u(\mathbf{r},t)\} = \int Z\{\eta\}\exp\left(\int u\eta\,\mathrm{d}\mathbf{r}\,\mathrm{d}t\right)\mathrm{D}\eta, \tag{6}$$

$$Z\{\eta(\mathbf{r},t)\} \equiv \left\langle\prod_{(\mathbf{r},t)}\delta\left\{\dot{\eta} + \frac{\delta V}{\delta\eta} - \zeta\right\}\det\left|\frac{\delta\zeta}{\delta\eta}\right|\right\rangle_0, \tag{7}$$

so that its variational derivatives with respect to an auxiliary field $u(\mathbf{r},t)$ give correlators of observables (see Eq.(72)). Obviously, $Z\{u\}$ represents the functional Laplace transfor-

mation of the dependence $Z\{\eta\}$, δ–function reflects the condition (3), the determinant is Jacobian of the integration variable change from ζ to η.

2.1 Fluctuation amplitude as a component of nilpotent field

Further development of the field scheme proceed depending on the type of connection between stochastic variable ζ and order parameter η. For thermodynamic system, where the thermostat state does not depend on η, the determinant in Eq.(7) assumes constant value that can be chosen as unity. Then, by using integral representation for δ–function

$$\delta\{x(\mathbf{r},t)\} = \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \exp\left(-\int \varphi x \mathrm{d}\mathbf{r}\mathrm{d}t\right) D\varphi \tag{8}$$

with the ghost field $\varphi(\mathbf{r},t)$ and averaging over distribution (5), we have the functional (7) in the standard form

$$Z\{\eta(\mathbf{r},t)\} = \int \exp\left[-S\{\eta(\mathbf{r},t),\varphi(\mathbf{r},t)\}\right] D\varphi, \tag{9}$$

where the action $S = \int \mathcal{L}\mathrm{d}\mathbf{r}\mathrm{d}t$ is measured in units $S_s = \gamma^2(T/D)^3$ with the Lagrangian given by

$$L(\eta,\varphi) = (\varphi\dot{\eta} - \varphi^2/2) + \varphi(\delta V/\delta\eta). \tag{10}$$

In order to obtain a canonical form of the Lagrangian (10) let us introduce the nilpotent field

$$\phi_\varphi = \eta + \vartheta\varphi \tag{11}$$

with Bose components η, φ, and nilpotent coordinate ϑ obeys the relations

$$\vartheta^2 = 0, \qquad \int \mathrm{d}\vartheta = 0, \qquad \int \vartheta \mathrm{d}\vartheta = 1. \tag{12}$$

As is shown in Appendix A, the first bracketed expression in Lagrangian (10) takes the form of kinetic energy in the Dirac field scheme [8]:

$$\kappa = \frac{1}{2}\int \phi D\phi \mathrm{d}\vartheta. \tag{13a}$$

Hereafter indexes are suppressed. The Hermite operator D is defined by equality

$$D_\varphi = -\frac{\partial}{\partial\vartheta} + \left(1 - 2\vartheta\frac{\partial}{\partial\vartheta}\right)\frac{\partial}{\partial t} \tag{14}$$

and enjoys the property (A.6). On the other hand, the algebraic properties (12) of coordinate ϑ allow to rewrite the last term in Eq.(10) in the standard form of potential energy (see Appendix A)

$$\pi = \int V(\phi)\mathrm{d}\vartheta. \tag{13b}$$

The resulting Lagrangian (10) of the Euclidean field theory is

$$L \equiv \kappa + \pi = \int \lambda \ \mathrm{d}\vartheta, \quad \lambda(\phi) \equiv \frac{1}{2}\phi D\phi + V(\phi). \tag{15}$$

According to Appendix A, the expressions (10), (15) become invariant with respect to transformation $e^{\varepsilon D}$ given by operator (14) if only a parameter $\varepsilon \to 0$ is pure imaginary and the fields $\eta(\mathbf{r},t)$, $\varphi(\mathbf{r},t)$ are complex–valued. Then, operator D is the generator of the nilpotent group.

After equating the first variation of the functional

$$s\{\phi(\zeta)\} = \int \lambda(\phi(\zeta))\mathrm{d}\zeta, \qquad \zeta \equiv \{\mathbf{r}, t, \vartheta\} \tag{16}$$

to zero, we obtain the Euler–Lagrange equation

$$D\frac{\delta\lambda}{\delta D\phi} + \frac{\delta\lambda}{\delta\phi} = 0, \tag{17}$$

Substituting the expression (15) in Eq.(17) yields the equation of motion

$$D\phi + \delta V/\delta\phi = 0. \tag{18}$$

Projection along axes of usual and nilpotent variables gives the system of the equations

$$\dot{\eta} = -\delta V/\delta\eta + \varphi, \tag{19}$$

$$\dot{\varphi} = \frac{\delta^2 V}{\delta^2 \eta} \ \varphi, \tag{20}$$

that determines kinetics of the phase transition. Being obtained from the extremum condition for Lagrangian (10) these equations determine the maximum value of the probability distribution

$$P\left\{\eta(\mathbf{r},t), \varphi(\mathbf{r},t)\right\} = Z^{-1}\exp\left(-\int L(\eta,\varphi)\mathrm{d}\mathbf{r}\mathrm{d}t\right), \tag{21}$$

that specifies the partition function $Z \equiv Z\{u=0\}$ in Eq.(6). Comparison of expression (19) with Langevin equation (3) leads to the conclusion that the quantity φ determines the most probable value of fluctuation of the field conjugated to the order parameter. On the other hand, it means that the initial one–modal distribution (5) transforms into the final two–modal form (21).

2.2 Conjugate field as a component of nilpotent field

It is well to bear in mind that there is another representation of two–component nilpotent field. Let us introduce field $f(\mathbf{r},t)$ defined by the relation

$$\dot{\eta} = f + \varphi. \tag{22}$$

Then the Lagrangian (10) takes the form

$$L(\eta, f) = \frac{1}{2}\left(\dot{\eta}^2 - f^2\right) - \frac{\delta V}{\delta \eta} f + \frac{\delta V}{\delta \eta}\dot{\eta}. \tag{23}$$

Since the last term of Eq.(23) is the total derivative of V with respect to time, its contribution to the partition function gives a factor that is integral over initial and final fields $\eta_i(\mathbf{r}) \equiv \eta(\mathbf{r}, t_i)$, $\eta_f(\mathbf{r}) \equiv \eta(\mathbf{r}, t_f)$ (here we return to dimensional magnitude of the potential V).

$$Z = \int \exp\left(-\frac{V\{\eta_f\} - V\{\eta_i\}}{T}\right) D\eta_i D\eta_f \tag{24}$$

The remaining part of Lagrangian (23) yields the Euler equations

$$\ddot{\eta} = -\frac{\delta^2 V}{\delta^2 \eta}\ f, \tag{25}$$

$$f = -\delta V/\delta\eta. \tag{26}$$

Differentiating Eq.(19) with respect to time and taking into account Eqs.(20), (22), it is not difficult to derive Eq.(25). As for Eq.(26), it defines $f(\mathbf{r},t)$ as the field conjugated to the order parameter $\eta(\mathbf{r},t)$. Note that Eq.(26) implies the force f explicitly does not depend on the time t.

By analogy with the definition (11) let us introduce now another nilpotent field [9]

$$\phi_f = \eta - \vartheta f, \tag{27}$$

where Bose components are the order parameter η and the force f with opposite sign. As it is shown in Appendix A, expression for Lagrangian in terms of ϕ_f has the same form as in Eq.(15) with the generator of the nilpotent group given by

$$D_f = -\left(\frac{\partial}{\partial\vartheta} + \vartheta\frac{\partial^2}{\partial t^2}\right). \tag{28}$$

Note that D_f obeys the algebraic relation (A.6).

2.3 Connection between two–component nilpotent representations

In this subsection we discuss the relation between the two above two–component nilpotent fields (11) and (27) that makes using of the fields algebraically equivalent.

Let us introduce the operators $\tau_\pm = e^{\pm\vartheta\partial_t}$, $\partial_t \equiv \partial/\partial t$ that induce the following transformations of the fields ϕ_φ and ϕ_f

$$\tau_\pm\phi_{\mp\varphi}(t) = \phi_{\mp f}(t), \qquad \tau_\pm\phi_{\pm f}(t) = \phi_{\pm\varphi}(t). \tag{29}$$

Eq.(29) shows that operators $\tau_\pm$ transform the field to its counterpart. So we have the mappings relating the representations.

By making expansion in power series over ϑ, with help of Eqs.(12), (22) one obtaines

$$\phi_{\mp\varphi}(t \pm \vartheta) = \phi_{\mp f}(t), \qquad \phi_{\pm f}(t \pm \vartheta) = \phi_{\pm\varphi}(t), \tag{30}$$

that shows that operators $\tau_\pm$ shift the physical time t by the nilpotent values $\pm\vartheta$:

$$\tau_\pm\phi_{\mp\varphi}(t) = \phi_{\mp\varphi}(t \pm \vartheta), \qquad \tau_\pm\phi_{\pm f}(t) = \phi_{\pm f}(t \pm \vartheta). \tag{31}$$

The same results can be obtained by using matrix representation defined by Eqs. (A.7), (A.9) and (A.10).

On the other hand, the above mappings $\tau_+\phi_f = \phi_\varphi$, $\tau_-\phi_\varphi = \phi_f$ induce corresponding transformations of the generators (14), (28)

$$D_f = \tau_- D_\varphi \tau_+, \qquad D_\varphi = \tau_+ D_f \tau_-. \tag{32}$$

Note that the action with Lagrangian (15) is covariant with respect to transformations (29), provided $\dot{f} \equiv 0$ (potential V does not depend on time explicitly).

3 Reduction of proper SUSY fields to the two–component forms

The considerations given in previous section rest on the assumption that the Jacobian of variable change from ζ to the order parameter η is constant. However, in general case determinant of an arbitrary matrix $|A|$ can be expressed as an integral over Grassmann conjugate fields $\psi(\mathbf{r}, t)$, $\overline{\psi}(\mathbf{r}, t)$, that meet conditions type of Eqs.(12)

$$\det|A| = \int \exp\left(\overline{\psi}A\psi\right) \mathrm{d}^2\psi, \qquad \mathrm{d}^2\psi = \mathrm{d}\psi\mathrm{d}\overline{\psi}. \tag{33}$$

Physically, the appearance of new degrees of freedom ψ, $\overline{\psi}$ means that the state of thermostat turns out to be dependent on the order parameter — as it is inherent in self–organized

system [10]. As a result, the Lagrangian (10) supplemented with the Grassmann fields ψ, $\overline{\psi}$ takes the form

$$\mathcal{L}(\eta,\varphi,\psi,\overline{\psi}) = \left(\varphi\dot{\eta} - \frac{\varphi^2}{2} + \frac{\delta V}{\delta\eta}\varphi\right) - \overline{\psi}\left(\frac{\partial}{\partial t} + \frac{\delta^2 V}{\delta\eta^2}\right)\psi. \tag{34}$$

Introducing the four–component SUSY field

$$\Phi_\varphi = \eta + \overline{\theta}\psi + \overline{\psi}\theta + \overline{\theta}\theta\varphi, \tag{35}$$

by analogy with previous section the SUSY Lagrangian is

$$\mathcal{L} = \int \Lambda \mathrm{d}^2\theta, \qquad \Lambda(\Phi_\varphi) \equiv \frac{1}{2}(\overline{\mathcal{D}}_\varphi\Phi_\varphi)\,(\mathcal{D}_\varphi\Phi_\varphi) + V(\Phi_\varphi), \qquad \mathrm{d}^2\theta \equiv \mathrm{d}\theta\mathrm{d}\overline{\theta}, \tag{36}$$

where θ, $\overline{\theta}$ are Grassmann conjugate coordinates that replace the nilpotent one ϑ. As compared with Eq.(15), where the kernel λ is linear in the generator (14), a couple of the Grassmann non–conjugated operators

$$\mathcal{D}_\varphi = \frac{\partial}{\partial\overline{\theta}} - 2\theta\frac{\partial}{\partial t}, \qquad \overline{\mathcal{D}}_\varphi = \frac{\partial}{\partial\theta} \tag{37}$$

enters the expression for SUSY Lagrangian. The Euler equation for SUSY action reads

$$-\frac{1}{2}[\overline{\mathcal{D}},\mathcal{D}]\Phi + \frac{\delta V}{\delta\Phi} = 0, \tag{38}$$

where the square brackets denote the commutator. Projection of Eq.(38) along the SUSY axes 1, $\overline{\theta}$, θ, $\overline{\theta}\theta$ gives the equations of motion

$$\dot{\eta} - \nabla^2\eta = -\partial V/\partial\eta + \varphi, \tag{39a}$$

$$\dot{\varphi} + \nabla^2\varphi = (\partial^2 V/\partial\eta^2)\varphi - (\partial^3 V/\partial\eta^3)\overline{\psi}\psi, \tag{39b}$$

$$\dot{\psi} - \nabla^2\psi = -(\partial^2 V/\partial\eta^2)\psi, \tag{39c}$$

$$-\dot{\overline{\psi}} - \nabla^2\overline{\psi} = -(\partial^2 V/\partial\eta^2)\overline{\psi}, \tag{39d}$$

that give Eqs.(19), (20) at $\psi = \overline{\psi} = 0$. It can be readily shown that this system can be obtained from the Lagrangian (34). From Eqs.(39c) and (39d) we obtain the conservation law $\dot{S} + \nabla\mathbf{j} = 0$ for the quantities

$$S = \overline{\psi}\psi, \qquad \mathbf{j} = (\nabla\overline{\psi})\psi - \overline{\psi}(\nabla\psi). \tag{40}$$

For inhomogeneous thermodynamic systems S is a density of sharp boundaries, $\mathbf{j}$ is a corresponding current [6]. In particular, the approach of the four–component SUSY field complies with the strong segregation limit requirement of copolymer theory [11]. For self–organized system the magnitude S gives the entropy, $\mathbf{j}$ is the probability current [10].

So, for thermodynamic system, where the entropy is conserved, we could disregard the Grassmann fields $\psi(\mathbf{r},t)$, $\overline{\psi}(\mathbf{r},t) = const$. As a result, the four–component SUSY field (35) is reduced to the two–component form (11).

In order to justify this let us write the kinetic term of the Lagrangian (36) in the form $-(1/4)\Phi_\varphi \left[\overline{\mathcal{D}}_\varphi, \mathcal{D}_\varphi\right] \Phi_\varphi$ where

$$-\frac{1}{2}\left[\overline{\mathcal{D}}_\varphi, \mathcal{D}_\varphi\right] = -\frac{\partial^2}{\partial\theta\partial\overline{\theta}} + \left(1 - 2\theta\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial t}. \tag{41}$$

The expression (41) restricted to two–component form with $\vartheta \equiv \overline{\theta}\theta$ yields the generator (14) as is needed. It is of interest to note that variable ϑ satisfies (12). In addition, since the self–conjugated value $\vartheta = \overline{\vartheta}$ is commutating quantity, it is nilpotent rather than Grassmannian.

As in the case of two–component nilpotent fields in Section II, one can go over from the fluctuation amplitude φ to the conjugate force f by using Eq.(22). Then, the first bracket in Lagrangian (34) takes the form (23) and instead of the system (39) one obtains the equation (cf. Eq.(25))

$$\ddot{\eta} = -(\delta^2 V/\delta\eta^2)f - (\delta^3 V/\delta\eta^3)\overline{\psi}\psi \tag{42}$$

supplemented with the definition of force (26) and the equations (39c,d) for the Grassmann fields $\psi(\mathbf{r},t)$, $\overline{\psi}(\mathbf{r},t)$. As above, the equation of motion (42) can be derive by differentiating Eq.(39a) with respect to time and taking into account Eqs.(22), (39b). The corresponding Lagrangian $\mathcal{L}(\eta, f, \psi, \overline{\psi})$ takes the SUSY form (cf. Eqs.(36))

$$\mathcal{L} = \int \Lambda \mathrm{d}^2\theta, \qquad \Lambda(\Phi_f) \equiv -\frac{1}{2}\Phi_f\overline{\mathcal{D}}_f\mathcal{D}_f\Phi_f + V(\Phi_f) \tag{43}$$

with the SUSY field (cf. Eq.(27))

$$\begin{aligned} &\Phi_f = \eta + \overline{\theta}\psi + \overline{\psi}\theta - \overline{\theta}\theta f \equiv \Phi_\varphi - \overline{\theta}\theta\dot{\Phi}_f = T_-\Phi_\varphi, \\ &T_\pm \equiv e^{\pm\overline{\theta}\theta\partial_t}, \quad \partial_t \equiv \partial/\partial t \end{aligned} \tag{44}$$

and the Grassmann conjugated operators (cf. Eqs.(37))

$$\mathcal{D}_f = \frac{\partial}{\partial\overline{\theta}} - \theta\frac{\partial}{\partial t}, \qquad \overline{\mathcal{D}}_f = \frac{\partial}{\partial\theta} - \overline{\theta}\frac{\partial}{\partial t}. \tag{45}$$

By analogy with Eqs.(29)–(31) with operators $\tau_\pm = e^{\pm\vartheta\partial_t}$ replaced by $T_\pm \equiv e^{\pm\overline{\theta}\theta\partial_t}$, where $\partial_t \equiv \partial/\partial t$, the SUSY fields (35), (44) can be transformed into each other and couples of the SUSY operators (37), (45) are related by means of transformations (cf. Eqs.(32)):

$$\mathcal{D}_f = T_-\mathcal{D}_\varphi T_+, \qquad \overline{\mathcal{D}}_f = T_-\overline{\mathcal{D}}_\varphi T_+. \tag{46}$$

According to Eq.(45), kernel of kinetic part of the SUSY Lagrangian (43) is (cf. Eq.(41))

$$-\overline{\mathcal{D}}_f\mathcal{D}_f = -\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\overline{\theta}} + \overline{\theta}\theta\frac{\partial^2}{\partial t^2}\right) + \left(\overline{\theta}\frac{\partial}{\partial\overline{\theta}} - \theta\frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial t}. \tag{47}$$

Note that the operator (28) can be obtained from Eq.(47) by taking into account the condition of the Fermion number conservation $\overline{\theta}(\partial/\partial\overline{\theta}) = \theta(\partial/\partial\theta)$ and by setting $\overline{\theta}\theta$ equal to ϑ.

So, both four–component Grassmann fields (35), (44) with SUSY generators given by Eqs.(37), (45) can be reduced to the corresponding two–component fields, Eqs.(11), (27), with operators (14), (28), respectively.

It is worthwhile to mention that such reduction can be obtained according to the SUSY gauge conditions

$$\mathcal{D}\Phi = 0; \qquad \overline{\mathcal{D}}\Phi = 0. \tag{48}$$

Indeed, according to definitions (35), (37), (44), (45) the equalities (48) give the relation

$$\overline{\theta}\psi + \overline{\psi}\theta - 2\overline{\theta}\theta f = 0, \tag{49}$$

that reduces the SUSY field (44) to the form (27) with opposite sign before f, provided $\vartheta \equiv \overline{\theta}\theta$.

Despite of the same number of components, one has to have in mind that the reduced SUSY field from Eq.(27) and couple of Grassmann conjugate chiral SUSY fields (B.9), which appearance is a consequence of SUSY gauge invariance also (see Appendix B), have different physical meaning. The main distinction is that the first field consists of two Bose components η, f, whereas the chiral SUSY fields ϕ_+, ϕ_- are the combinations of Bose η and Fermi ψ, $\overline{\psi}$ components. Formally, this is due to the fact that for separation of the chiral SUSY fields the conditions (B.7) of the SUSY gauge invariance are fulfilled not for the initial SUSY field Φ, which satisfies to conditions (48), but for components $\Phi_\pm$, resulting from Φ under the action of operators $T_\pm = \exp\left(\pm\overline{\theta}\theta\partial_t\right)$ (see Eq.(B.1)).

According to the above considerations, the transformation operators $T_\pm$, that shift physical time t by Grassmann values $\pm\overline{\theta}\theta$, relate the SUSY fields (35), (44) and corresponding generators (37), (45). It should be emphasized that only the latter form a pair of Grassmann conjugated operators. The physical reason of this symmetry is that the corresponding equation of (42) is invariant with respect to the time inversion, whereas the equations (39a), (39b) for components of the SUSY field (35) are not. However,in addition to the field $\Phi_\varphi \equiv \Phi_+$ obtained from the initial field Φ_f under the action of operator T_+, another SUSY field Φ_- emerge under the action of operator T_- that shifts the time t in opposite direction. From Eqs.(B.5), (22) it can be seen that the fields $\Phi_\pm \equiv \Phi_\varphi(\pm t)$ correspond to opposite directions of time.

However, equations (39c), (39d) for the Grassmann components $\psi(\mathbf{r},t)$, $\overline{\psi}(\mathbf{r},t)$ are invariant under the action of $T_{\pm}$ To break the invariance let us introduce additional operators of transformation

$$\widetilde{T}_{\pm} = \exp\left[\varepsilon\left(\delta_{\pm}\overline{\theta}\psi + \delta_{\mp}\overline{\psi}\theta\right)\right] \tag{50}$$

where source parameter $\varepsilon \to 0$; $\delta_{+} = 1$, $\delta_{-} = 0$ for the positive time direction and $\delta_{+} = 0$, $\delta_{-} = 1$ otherwise. The Euler SUSY equation (38) for transformed superfield $\widetilde{\Phi}_{\pm} \equiv \widetilde{T}_{\pm}\Phi_{\varphi}$ is reduced to the components

$$\dot{\eta} - \nabla^2\eta = -\partial V/\partial\eta + \varphi - \varepsilon\overline{\psi}\psi, \tag{51a}$$

$$\dot{\varphi} + \nabla^2\varphi = (\partial^2 V/\partial\eta^2)\varphi - (\partial^3 V/\partial\eta^3)\overline{\psi}\psi + \varepsilon\overline{\psi}\dot{\psi}, \tag{51b}$$

$$\dot{\psi} - \nabla^2\psi = -(\partial^2 V/\partial\eta^2)\psi - \varepsilon\left\{\delta_{-}(\dot{\psi}/\psi)\eta + \delta_{+}\left[(\dot{\eta} - \varphi) + (\partial^2 V/\partial\eta^2)\eta\right]\right\}\psi, \tag{51c}$$

$$\dot{\overline{\psi}} + \nabla^2\overline{\psi} = (\partial^2 V/\partial\eta^2)\overline{\psi} - \varepsilon\left\{\delta_{+}(\dot{\overline{\psi}}/\overline{\psi})\eta - \delta_{-}\left[(\dot{\eta} - \varphi) + (\partial^2 V/\partial\eta^2)\eta\right]\right\}\overline{\psi}, \tag{51d}$$

where the terms of first order ε are kept. These equations give Eqs.(39) at $\varepsilon \to 0$, but combination of Eqs.(51c), (51d) at $\varepsilon \neq 0$ leads to the following equation for the quantities (40)

$$\dot{S} + \nabla\mathbf{j} = \pm\varepsilon F S, \quad F \equiv \partial V/\partial\eta - 2(\partial^2 V/\partial\eta^2)\eta \tag{52}$$

instead of the law of entropy conservation. Since entropy S of a closed system ($\nabla\mathbf{j} = 0$) increases in time, provided $F > 0$, in Eq.(52) one has to choose the upper sign corresponding to the positive time direction. So, the operator (50) breaks symmetry with respect to the time reversibility. The above–mentioned condition of positiveness for effective force $F \equiv \partial V/\partial\eta - 2(\partial^2 V/\partial\eta^2)\eta$ means that the effective potential V is an increasing convex function of the η that is inherent in an unstable system. It is of interest to note that near the equilibrium state, where $\partial V/\partial\eta = 0$, $\eta \ll 1$, the force $F \simeq -(\partial^2 V/\partial\eta^2)\eta$ is always positive for unstable system.

Finally, in order to visualize the difference between two–component nilpotent fields (11), (27) and chiral fields (B.9) let us represent the SUSY field (44) as a vector in four–dimensional space with axes $\theta^0 = \overline{\theta}^0 \equiv 1$, $\overline{\theta}$, θ, $\overline{\theta}\theta \equiv \vartheta$. Then conditions (48) of the SUSY gauge invariance mean that field (44) is reduced to the vector (27) belonging to a plane formed by axes 1, ϑ. Accordingly, the conditions (B.7) of the chiral gauge invariance split total SUSY space into a couple of orthogonal subspaces, the first of which has the axes 1, θ and contains the vector ϕ_{-}, and second — axes 1, $\overline{\theta}$ and vector ϕ_{+}. Since these subspaces are Grassmann conjugated, $\overline{\phi}_{-} = \phi_{+}$, it is enough to use one of them, considering either vector ϕ_{-}, or ϕ_{+} (see Appendix B). Such program was realized in Ref.[12], whereas the above used nilpotent field (27) is derived by projecting chiral vectors $\phi_{\pm}$ to a plane formed by axes 1, ϑ. It follows that our approach stated on the using nilpotent fields (11), (27)

and the theory [12] are equivalent. The SUSY method presented in the book [13] is also based on usage of the chiral fields $\phi_- = \varphi - i\overline{\psi}\theta$, $\phi_+ = \eta + \overline{\theta}\psi$ (cf. with (B.9)) that contain the fluctuation φ as a Bose component of the field ϕ_- and the order parameter η in field ϕ_+.

4 SUSY correlation techniques

In this section correlators of the proper SUSY fields (35), (44) will be studied. It will be shown how the relevant correlation techniques can be reduced to the simplest scheme by making use of the two–component field (11).

To begin with let us introduce the SUSY correlator

$$C(z, z') = \langle \Phi(z)\Phi(z') \rangle, \qquad z \equiv \{\mathbf{r}, t, \overline{\theta}, \theta\}. \tag{53}$$

From the equation of motion (38) we have the equation for Fourier transform of the bare SUSY correlator $C^{(0)}(z, z')$ with the potential $V_0 = (1/2)\Phi^2$ in the following form

$$L_{\mathbf{k}\omega}(\theta) C^{(0)}_{\mathbf{k}\omega}(\theta, \theta') = \delta(\theta, \theta'), \qquad L \equiv 1 - (1/2)[\overline{\mathcal{D}}, \mathcal{D}], \tag{54}$$

where $\delta(\theta, \theta')$ is the Grassmann δ–function

$$\delta(\theta, \theta') = (\overline{\theta} - \overline{\theta}')(\theta - \theta'), \tag{55}$$

ω is the frequency and $\mathbf{k}$ is the wave vector. The solution of Eq.(54) reads

$$C^{(0)}(\theta, \theta') = \frac{\left(1 + (1/2)[\overline{\mathcal{D}}, \mathcal{D}]\right) \delta(\theta, \theta')}{1 - (1/4)[\overline{\mathcal{D}}, \mathcal{D}]^2}, \tag{56}$$

where the indexes ω, $\mathbf{k}$ are suppressed for brevity. From the definitions (37), (55) and equality $[\overline{\mathcal{D}}, \mathcal{D}]^2 = -4\omega^2$ (see Eqs.(A.11)), the bare SUSY correlator for SUSY field (35) can be written in the explicit form

$$C^{(0)}_{\varphi}(\theta, \theta') = \frac{1 + (1 - \mathrm{i}\omega)(\overline{\theta} - \overline{\theta}')\theta - (1 + \mathrm{i}\omega)(\overline{\theta} - \overline{\theta}')\theta'}{1 + \omega^2}. \tag{57}$$

In the case of the SUSY field (44), by using transformation (46) the above result is found to be modified by adding the term $\mathrm{i}\omega(\overline{\theta}\theta - \overline{\theta}'\theta')$ to the numerator of Eq.(57).

It is convenient to introduce the following components as a basis for expansion of SUSY correlators

$$T(\theta,\theta')=1,\qquad B_0(\theta,\theta')=\overline{\theta}\theta,\qquad B_1(\theta,\theta')=\overline{\theta}'\theta', \tag{58}$$
$$F_0(\theta,\theta')=-\overline{\theta}'\theta,\qquad F_1(\theta,\theta')=-\overline{\theta}\theta'.$$

Let us define the operator product

$$X(\theta,\theta')=\int Y(\theta,\theta'')Z(\theta'',\theta')\mathrm{d}^2\theta'' \tag{59}$$

for superspace functions Y, Z. Eq.(59) immediately provide the multiplication rules for the basis operators (58) summarized in Table I:

Table I

$l\backslash r$	$\mathbf{T}$	$\mathbf{B}_0$	$\mathbf{B}_1$	$\mathbf{F}_0$	$\mathbf{F}_1$
$\mathbf{T}$	0	$\mathbf{T}$	0	0	0
$\mathbf{B}_0$	0	$\mathbf{B}_0$	0	0	0
$\mathbf{B}_1$	$\mathbf{T}$	0	$\mathbf{B}_1$	0	0
$\mathbf{F}_0$	0	0	0	$\mathbf{F}_0$	0
$\mathbf{F}_1$	0	0	0	0	$\mathbf{F}_1$

The operators $\mathbf{T}$, $\mathbf{B}_{0,1}$, $\mathbf{F}_{0,1}$ then form the closed basis, so that expansions for correlators are (see Eqs.(C.7), (C.10))

$$\mathbf{C}_\varphi=S\mathbf{T}+G_+\left(\mathbf{B}_0+\mathbf{F}_0\right)+G_-\left(\mathbf{B}_1+\mathbf{F}_1\right), \tag{60}$$
$$\mathbf{C}_f=S\mathbf{T}+m_+\mathbf{B}_0+m_-\mathbf{B}_1+G_+\mathbf{F}_0+G_-\mathbf{F}_1$$

where in accordance with Ward identity (C.6) corresponding to the first generator (C.5) term proportional to $\overline{\theta}\theta\overline{\theta}'\theta'$ is dropped. Inserting SUSY fields (35), (44) into Eq.(53) provides the coefficients of expansions (60) (cf. Eqs.(C.8), (C.11)):

$$S=\langle|\eta|^2\rangle;\quad m_+=\langle\eta^*\rangle f_{\mathrm{ext}},\quad m_-=\langle\eta\rangle f^*_{\mathrm{ext}},\quad f_{\mathrm{ext}}\equiv -f; \tag{61}$$
$$G_+=\langle\varphi\eta^*\rangle=\langle\overline{\psi}\psi^*\rangle,\quad G_-=\langle\eta\varphi^*\rangle=\langle\overline{\psi}^*\psi\rangle.$$

So, quantity S is the autocorrelator of order parameter η and magnitudes $m_\mp$ meet the condition $m_+^*=m_-$ and determine the averaged order parameter $\langle\eta\rangle$ corresponding to external force $f_{\mathrm{ext}}\equiv -f$. The retarded and advanced Green functions $G_\mp$ give the response of order parameter η to fluctuation amplitude φ and *vice versa* (moreover, functions $G_\pm$ determine correlation of the Grassmann fields $\overline{\psi}$, ψ). As it is known [14], the Fourier transforms $G_\mp(\omega)$ of retarded and advanced Green functions are analytical in upper and

lower half–planes of complex frequency ω with cut along real axis ω'. There is the jump $G_-(\omega') - G_+(\omega') = 4\mathrm{i}\,\mathrm{Im}\,G_-(\omega')$, so that the relations (C.9), (C.12) assume the usual form of the fluctuation–dissipation theorem:

$$G_\pm(\omega) = m_\pm(\omega) \mp \mathrm{i}\omega S(\omega), \quad S(\omega') = (2/\omega')\,\mathrm{Im}\,G_-(\omega') \tag{62}$$

where the frequency ω' is real. The expression for bare correlator (57) gives:

$$S^{(0)} = m_\pm^{(0)} = (1+\omega^2)^{-1}, \qquad G_\pm^{(0)} = (1 \pm \mathrm{i}\omega)^{-1}. \tag{63}$$

Integrate the last equation of (62) and taking into account the spectral representation

$$C(\omega) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\pi} \frac{\mathrm{Im}\,C(\omega')}{\omega' - \omega}. \tag{64}$$

we arrive at useful relation

$$S(t=0) = G_\pm(\omega = 0) \equiv \chi, \tag{65}$$

where the last identity is the definition of susceptibility χ.

The expansions (60) make it possible to handle the SUSY correlator (53) as a vector of space constructed as the direct product of the SUSY fields (35) or (44). The representation (35) is of special convenience because it allows using of reduced basis

$$\mathbf{A} \equiv \mathbf{B}_0 + \mathbf{F}_0, \qquad \mathbf{B} \equiv \mathbf{B}_1 + \mathbf{F}_1. \tag{66}$$

Along with $\mathbf{T}$, they form more compact basis and obey the following multiplication rules:

Table II

$l\backslash r$	$\mathbf{T}$	$\mathbf{A}$	$\mathbf{B}$
$\mathbf{T}$	0	$\mathbf{T}$	0
$\mathbf{A}$	0	$\mathbf{A}$	0
$\mathbf{B}$	$\mathbf{T}$	0	$\mathbf{B}$

The expansion Eqs.(60) then takes the form

$$\mathbf{C}_\varphi = S\mathbf{T} + G_+\mathbf{A} + G_-\mathbf{B}. \tag{67}$$

So, using Ward identities allows to get rid of autocorrelators of the Grassmann fields ψ, $\overline{\psi}$ (see relations (61)). As a result, there are three basic correlators: the advanced

and retarded Green functions $G_{\pm}$ and structure factor S. They yield the most compact expansion (67) for arbitrary SUSY correlator of fields (35). It is ready to show that expansion of the same form can be obtained on the basis of the two–component field (11) representation. Indeed, in this case by comparison between equations of motion (18) and (38) the commutator $-(1/2)[\overline{\mathcal{D}}, \mathcal{D}]$ in expression (56) should be replaced by generator (14) and nilpotent δ–function should be $\delta(\vartheta - \vartheta') = \vartheta + \vartheta'$. So the resulting bare correlator is

$$C^{(0)}(\vartheta, \vartheta') = \frac{1 + (1 - \mathrm{i}\omega)\vartheta + (1 + \mathrm{i}\omega)\vartheta'}{1 + \omega^2} \tag{68}$$

instead of Eq.(57). It is easily to see that using the definitions (cf. Eqs.(58))

$$T(\vartheta, \vartheta') = 1, \quad A(\vartheta, \vartheta') = \vartheta, \quad B(\vartheta, \vartheta') = \vartheta' \tag{69}$$

gives the relevant expansion (67). As a result, in what follows we can use two–component field (11).

In particular, for inverse of the SUSY correlator (67) we have

$$\mathbf{C}^{-1} = -G_{+}^{-1} S G_{-}^{-1} \mathbf{T} + G_{+}^{-1} \mathbf{A} + G_{-}^{-1} \mathbf{B}. \tag{70}$$

It is worthwhile to note that according to definitions (66), (58) the basis operators $\mathbf{A} \equiv \mathbf{B} \equiv 0$ provided $\theta = \theta'$, so that $C(\theta, \theta) = C(\vartheta, \vartheta) = S$ and

$$\begin{aligned} \int C(z, z) \mathrm{d}z &= \int S(\mathbf{r}, t; \mathbf{r}, t) \mathrm{d}\mathbf{r} \mathrm{d}t \mathrm{d}^2\theta = 0, \\ \int C(\zeta, \zeta) \mathrm{d}\zeta &= \int S(\mathbf{r}, t; \mathbf{r}, t) \mathrm{d}\mathbf{r} \mathrm{d}t \mathrm{d}\vartheta = 0 \end{aligned} \tag{71}$$

where z, ζ are sets of variables $\{\mathbf{r}, t, \overline{\theta}, \theta\}$, $\{\mathbf{r}, t, \vartheta\}$, respectively. In the diagrammatic representation identities (71) imply the absence of bubble graphs contribution. The latter considerably reduces the number of graphs contributing to expansion of the perturbation theory (see below).

5 SUSY Perturbation Theory

Let us begin with the formula

$$C(\zeta, \zeta') = \left. \frac{\delta^2 Z\{u(\zeta)\}}{\delta u(\zeta) \delta u(\zeta')} \right|_{u=0}, \tag{72}$$

where generating functional (see Eqs.(6), (9))

$$Z\{u\} = \left\langle \exp\left(\int \phi u \mathrm{d}\zeta \right) \right\rangle \tag{73}$$

has the form of average over distribution (cf. Eq.(21))

$$P\{\phi\} = Z^{-1} \exp\left(-S\{\phi\}\right), \qquad S\{\phi\} = \int \lambda(\phi) \mathrm{d}\zeta, \tag{74}$$

with the Lagrangian λ defined by Eq.(15). In the zero–order approximation the action is quadratic

$$S_0 = \frac{1}{2} \int \phi L \phi \mathrm{d}\zeta, \qquad L \equiv 1 + D, \tag{75}$$

where generator D is given by Eq.(14). Corresponding distribution takes the SUSY Gaussian form (cf. Eq.(5))

$$P_0\{\phi\} = \left(\frac{\det|L|}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2} \int \phi L \phi \mathrm{d}\zeta\right\}. \tag{76}$$

So for the bare supercorrelator we have the expression

$$C^{(0)}(\zeta, \zeta') = L^{-1} \delta(\zeta, \zeta'), \quad \delta(\vartheta, \vartheta') \equiv \vartheta + \vartheta', \tag{77}$$

that leads to Eq.(56) with $-(1/2)[\overline{\mathcal{D}}, \mathcal{D}]$ replaced by D if Eqs.(75), (14) are taken into account. The linear operator $\mathbf{L} \equiv (\mathbf{C}^{(0)})^{-1}$ in accordance with Eq.(70) takes the form:

$$\begin{aligned} &\mathbf{L} = L\mathbf{T} + L_+\mathbf{A} + L_-\mathbf{B}; \\ &L = -1, \quad L_\pm = 1 \pm i\omega. \end{aligned} \tag{78}$$

To proceed, one need to separate out anharmonic part $S_1\{\phi\}$ of exponent in distribution (74) as a perturbation and to make expansion in power series over S_1. Insertion of this series into Eq.(72) gives

$$\mathbf{C}(\zeta, \zeta') = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \phi(\zeta) \left(S_1\{\phi\}\right)^n \phi(\zeta') \right\rangle_0, \tag{79}$$

where subscript "0" means averaging over the bare distribution (76). Further one has to make factorization by making use of the Wick theorem. Then within the n–th order of perturbation theory the expression (79) takes the form

$$C^{(n)}(\zeta, \zeta') = \iint C^{(0)}(\zeta, \zeta_1) \Sigma^{(n)}(\zeta_1, \zeta_2) C^{(0)}(\zeta_2, \zeta') \mathrm{d}\zeta_1 \, \mathrm{d}\zeta_2, \tag{80}$$

where $\Sigma^{(n)}(\zeta_1, \zeta_2)$ is the SUSY self–energy function of n–th order that should be calculated. The result essentially depends on the form of $V_1(\phi)$ that describes self–action effects. In what follows we will analyse two widely used models.

5.1 ϕ^4–model

Let the self–action potential be defined by the quartic dependence

$$V_1(\zeta) = \frac{\lambda}{4!}\phi^4(\zeta), \qquad \zeta = \{\mathbf{r}, t, \vartheta\} \tag{81}$$

with the anharmonicity constant $\lambda > 0$. Then terms of the first and second orders of series (79) are

$$C^{(1)}(\zeta, \zeta') = -\frac{\lambda}{4!}\int \left\langle \phi(\zeta)(\phi(\zeta_1))^4\phi(\zeta')\right\rangle_0 \mathrm{d}\zeta_1, \tag{82a}$$

$$C^{(2)}(\zeta, \zeta') = \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2 \iint \left\langle \phi(\zeta)(\phi(\zeta_1))^4(\phi(\zeta_2))^4\phi(\zeta')\right\rangle_0 \mathrm{d}\zeta_1\ \mathrm{d}\zeta_2\ . \tag{82b}$$

Now one has to count the number of possible pairings when using the Wick theorem. In Eq.(82a) the total number of pairings is 12, and the formula (82a) reads

$$C^{(1)}(\zeta, \zeta') = -\frac{\lambda}{2}\int C^{(0)}(\zeta, \zeta_1)C^{(0)}(\zeta_1, \zeta_1)C^{(0)}(\zeta_1, \zeta')\mathrm{d}\zeta_1 \equiv 0, \tag{83}$$

where Eqs.(53), (71) are taken into account. In Eq.(82b) the total number of pairings equals 192 and the Wick theorem gives

$$C^{(2)}(\zeta, \zeta') = \frac{\lambda^2}{6}\iint C^{(0)}(\zeta, \zeta_1)\left(C^{(0)}(\zeta_1, \zeta_2)\right)^3 C^{(0)}(\zeta_2, \zeta')\mathrm{d}\zeta_1\ \mathrm{d}\zeta_2. \tag{84}$$

Then, in accord with Eq.(80) the SUSY self–energy function in the second order of perturbation theory reads

$$\Sigma(\zeta, \zeta') = \frac{\lambda^2}{6}\left(C(\zeta, \zeta')\right)^3. \tag{85}$$

Here in terms of usual diagram ideology bare correlator is replaced by exact one.

In the diagrammatic representation terms (82a,b) correspond to the following graphs:

According to the rule (71) the former does not contribute to correlator, whereas the latter does (85).

By analogy with the SUSY correlator (67) it is convenient to expand the SUSY self–energy:

$$\Sigma = \Sigma\mathbf{T} + \Sigma_{+}\mathbf{A} + \Sigma_{-}\mathbf{B}. \tag{86}$$

To determine the coefficients $\Sigma_{\pm}$, Σ it should be taken into account that the multiplication rules in Eq.(85) differ from ones given by Table II. The reason is that Eq.(85) contains "element-to-element" products of nilpotent quantities [3] instead of the above operator product. Hence one has to use the multiplication rules given by the Table III:

Table III

$l\backslash r$	$T(\vartheta,\vartheta')$	$A(\vartheta,\vartheta')$	$B(\vartheta,\vartheta')$
$T(\vartheta,\vartheta')$	$T(\vartheta,\vartheta')$	$A(\vartheta,\vartheta')$	$B(\vartheta,\vartheta')$
$A(\vartheta,\vartheta')$	$A(\vartheta,\vartheta')$	0	0
$B(\vartheta,\vartheta')$	$B(\vartheta,\vartheta')$	0	0

As a result, the coefficients of expansion (86) take the form:

$$\Sigma(t) = (\lambda^2/6)S^3(t), \tag{87a}$$

$$\Sigma_{\pm}(t) = (\lambda^2/2)S^2(t)G_{\pm}(t). \tag{87b}$$

In the frequency representation that will be needed below we have

$$\Sigma(\omega) = \frac{\lambda^2}{6}\int\frac{\mathrm{d}\omega_1\,\mathrm{d}\omega_2}{(2\pi)^2}S(\omega-\omega_1-\omega_2)S(\omega_1)S(\omega_2), \tag{88a}$$

$$\Sigma_{\pm}(\omega) = \frac{\lambda^2}{2}\int\frac{\mathrm{d}\omega_1\,\mathrm{d}\omega_2}{(2\pi)^2}G_{\pm}(\omega-\omega_1-\omega_2)S(\omega_1)S(\omega_2). \tag{88b}$$

The obvious inconvenience of this expressions is the presence of convolutions. To get rid of them let us use the fluctuation–dissipation theorem

$$\Sigma(t=0) = \Sigma_{\pm}(\omega=0) \tag{89}$$

in the form of Eq.(65). Then from Eqs.(87a), (65) one obtains:

$$\Sigma_{\pm}(\omega=0) = (\lambda^2/6)\chi^3. \tag{90a}$$

5.2 Cubic anharmonicity

Apart from the ϕ^4–model studied above, there is a number of physical systems type of copolymers [2] where cubic anharmonicity

$$V_1(\zeta) = \frac{\mu}{3!}\phi^3(\zeta), \qquad \zeta = \{\mathbf{r}, t, \vartheta\} \tag{91}$$

has a dominant role (μ is the anharmonicity parameter). By analogy with Eqs.(82) it can be shown that the first non–vanishing contribution to the SUSY correlator (80) is

$$C^{(2)}(\zeta', \zeta') = \frac{1}{2!}\left(-\frac{\mu}{3!}\right)^2 \iint \left\langle \phi(\zeta)(\phi(\zeta_1))^3(\phi(\zeta_2))^3\phi(\zeta')\right\rangle_0 \mathrm{d}\zeta_1\ \mathrm{d}\zeta_2. \tag{92}$$

To facilitate the factorization of these products let us depict possible graphs of the second order in cubic anharmonicity μ:

The first of these graphs contains the bubble and does not contribute to the correlator. The contribution of the second graph is

$$\frac{\mu^2}{2}\iint C^{(0)}(\zeta, \zeta_1)\left(C^{(0)}(\zeta_1, \zeta_2)\right)^2 C^{(0)}(\zeta_2, \zeta')\mathrm{d}\zeta_1\ \mathrm{d}\zeta_2. \tag{93}$$

As a result, the SUSY self–energy function reads

$$\Sigma(\zeta, \zeta') = \frac{\mu^2}{2}\left(C(\zeta, \zeta')\right)^2, \tag{94}$$

where the bare SUSY correlators are replaced by exact ones. By using the multiplication rules from Table III the coefficients of the expansion (86) are derived

$$\Sigma(t) = (\mu^2/2)S^2(t), \tag{95a}$$

$$\Sigma_{\pm}(t) = \mu^2 S(t)G_{\pm}(t). \tag{95b}$$

These expressions, combined with Eqs.(87), determine the SUSY self-energy function completely. By analogy with Eq.(90a) we have the relation

$$\Sigma_{\pm}(\omega = 0) = \Sigma(t = 0) \equiv (\mu^2/2)\chi^2 \tag{90b}$$

Finally, the resulting expressions for coefficients of expansion (86) with both cubic and quartic anharmonicities included are

$$\Sigma(t) = \frac{1}{2}\left(\mu^2 + \frac{\lambda^2}{3}S(t)\right)S^2(t), \tag{96a}$$

$$\Sigma_{\pm}(t) = \left(\mu^2 + \frac{\lambda^2}{2}S(t)\right)S(t)G_{\pm}(t), \tag{96b}$$

$$\Sigma_{\pm}(\omega = 0) = \frac{1}{2}\left(\mu^2 + \frac{\lambda^2}{3}\chi\right)\chi^2 \tag{96c}$$

6 Self–consistent approach

6.1 Effective SUSY Lagrangian

Let us start with the total SUSY action taken in the site representation:

$$S = S_0 + S_1 + S_{int}; \tag{97}$$

$$S_0 \equiv \frac{1}{2}\sum_l \int \phi_l(t,\vartheta)\Big[1 + D(\vartheta)\Big]\phi_l(t,\vartheta)\mathrm{d}t\mathrm{d}\vartheta, \tag{97a}$$

$$S_1 \equiv \sum_l \int V_1\left(\phi_l(t,\vartheta)\right)\mathrm{d}t\mathrm{d}\vartheta, \tag{97b}$$

$$S_{int} \equiv \int\int V_{int}\left\{\phi_l(t,\vartheta),\phi_m(t',\vartheta')\right\}\delta(t-t')\mathrm{d}t\mathrm{d}t'\mathrm{d}\vartheta\mathrm{d}\vartheta', \qquad V_{int} \equiv V + W. \tag{97c}$$

where sites are labeled with l and the self–action term $V_1(\phi_l)$ (97b), that given by Eqs.(81) and (91), is separated out. The last term S_{int} describes the two–particle interaction V and the effective potential W is caused by averaging over quenched disorder. The potential V is assumed to be attractive and takes the standard form [15]

$$V = -\frac{1}{2}\sum_{lm} v_{lm}\phi_m(t,\vartheta)\phi_l(t',\vartheta')\phi_l(t',\vartheta')\phi_m(t,\vartheta) - \\ \frac{1}{2}\sum_{lm} v_{lm}\phi_l(t,\vartheta)\phi_l(t',\vartheta')\phi_m(t,\vartheta)\phi_m(t',\vartheta') \tag{98}$$

that, in the mean–field approximation, provides the following expression

$$V \simeq -\frac{v}{2}C(t,\vartheta;t,\vartheta)\sum_l \phi_l(t',\vartheta')\phi_l(t',\vartheta') - \frac{v}{2}C(t,\vartheta;t',\vartheta')\sum_l \phi_l(t,\vartheta)\phi_l(t',\vartheta'). \quad (99)$$

Hereafter $v \equiv \sum_m v_{lm} > 0$ is the interaction constant, $C(t,\vartheta;t',\vartheta') \equiv \langle \phi_m(t,\vartheta)\phi_m(t',\vartheta')\rangle$ is the SUSY correlator in the site representation. Averaging over quenched disorder in intersite couplings results in the effective attractive interaction [5]

$$W = -\frac{1}{2}\sum_{lm} w_{lm}\phi_l(t,\vartheta)\phi_l(t',\vartheta')\phi_m(t,\vartheta)\phi_m(t',\vartheta'). \quad (100)$$

By analogy with Eq.(99) it is supposed that

$$W \simeq -\frac{w}{2}C(t,\vartheta;t',\vartheta')\sum_l \phi_l(t,\vartheta)\phi_l(t',\vartheta'), \quad w \equiv \sum_m w_{lm} > 0. \quad (101)$$

So, the real interaction (99) contains both diagonal and non–diagonal in ϑ and ϑ' SUSY correlators, whereas the quenched disorder averaging results in non–diagonal expression (101) only. Obviously, within the framework of the replica approach such SUSY structure corresponds to the inter–replica overlapping that is responsible for the specific spin–glass behaviour [1].

Apart from the above contributions to SUSY action (97) it should be taken into account that the quenched disorder in force dispersion results in the additional interaction [5]

$$\Delta S_0 = \frac{h^2}{2}\sum_l \int \phi_{l\omega}(\vartheta)\delta(\omega)\phi_{l\omega}(\vartheta)\mathrm{d}\omega\mathrm{d}\vartheta. \quad (102)$$

where ω is the frequency and the intensity of the quenched disorder

$$h^2 = \frac{\overline{\left(f_l - \overline{f}\right)^2} - (\Delta\varphi)^2}{(\Delta\varphi)^2} \quad (103)$$

characterizes the site dispersion of the force f_l (overbar denotes the volume average), $(\Delta\varphi)^2 \equiv \varphi^2_{\omega=0}$ is mean–squared fluctuation of this force. Then, the mean–field SUSY action takes, in the site–frequency representation, the final form

$$S = \sum_l \int \lambda_{l\omega}(\vartheta)\frac{\mathrm{d}\omega}{2\pi}\mathrm{d}\vartheta + \sum_l \int \lambda_{l\omega}(\vartheta,\vartheta')\frac{\mathrm{d}\omega}{2\pi}\mathrm{d}\vartheta\mathrm{d}\vartheta' \quad (104)$$

with the SUSY Lagrangian

$$\lambda(\vartheta) \equiv \frac{1}{2}\phi(\vartheta)\left\{[1 + D(\vartheta)] + 2\pi h^2\delta(\omega) - vS\right\}\phi(\vartheta) + V_1(\phi(\vartheta)), \quad (105a)$$

$$\lambda(\vartheta,\vartheta') \equiv -\frac{1}{2}(v+w)\phi(\vartheta)C(\vartheta,\vartheta')\phi(\vartheta') \tag{105b}$$

where indexes l, ω are suppressed for brevity and the generator D is given by Eq.(14).

In the important case of random heteropolymer the interaction kernels are appeared to be of the form (98), (100), but indexes l, m denote wave vectors but site numbers (see [16]). So, in this case, the expressions (104), (105) can be modified by replacing site indexes by wave ones.

6.2 SUSY Dyson equation

The Dyson equation for the above SUSY Lagrangian is

$$\mathbf{C}^{-1} = \mathbf{L} - \Sigma - (v+w)\mathbf{C}. \tag{106}$$

Here $\mathbf{L}$ is defined by Eq.(78) where the first component is

$$L = L_0 + vS, \qquad L_0 = -(1 + 2\pi h^2 \delta(\omega)). \tag{107}$$

Projecting Eq.(106) along the "axes" (69), we come to the key equations written in the frequency representation

$$S = \frac{(\Sigma - L_0)G_+G_-}{1 - wG_+G_-}, \tag{108a}$$

$$G_\pm^{-1} + (v+w)G_\pm = L_\pm - \Sigma_\pm \tag{108b}$$

where Eq.(70) is used. These equations accompanied by Eqs.(96) for the components Σ, $\Sigma_\pm$ of the SUSY self–energy function form the closed system of equations for self–consistent analysis of non–equilibrium thermodynamic system.

7 Non–ergodicity and memory effects

As it is well–known, the memory is characterized by the Edwards–Anderson parameter [17]

$$q = \langle \eta(\infty)\eta(0) \rangle \tag{109}$$

that being late–time asymptotics of the correlator results in elongation of the structure factor:

$$S(t) = q + S_0(t) \tag{110}$$

where the component $S_0(t) \to 0$ at $t \to \infty$. By analogy with the elongated structure factor (110), the ergodicity breaking is allowed for by adding the term to the retarded Green function

$$G_-(\omega) = \Delta + G_{-0}(\omega). \tag{111}$$

The non–ergodicity parameter (irreversible response) in Eq.(111)

$$\Delta = \chi_0 - \chi \tag{112}$$

is determined by the adiabatic Cubo susceptibility $\chi_0 \equiv G_-(\omega = 0)$ and the thermodynamic one $\chi \equiv G_{-0}(\omega = 0)$.[2] If the latter is defined by the standard formula $\chi = \delta \langle \eta \rangle / \delta f_{ext}$ with external force $f_{ext} \equiv -f$, for determination of the former one has to use the correlation techniques discussed in Sect.IV.

To do this let us insert the elongated correlators (110), (111) into expressions (96). Then the renormalized components of the self–energy function take the form

$$\Sigma(t) = \frac{1}{2}\left(\mu^2 + \frac{\lambda^2}{3}q\right)q^2 + \left(\mu^2 + \frac{\lambda^2}{2}q\right)qS_0(t) + \Sigma_0(t),$$
$$\Sigma_0(t) \equiv \frac{1}{2}\left(\mu^2 + \lambda^2 q\right)S_0^2(t) + \frac{\lambda^2}{6}S_0^3(t); \tag{113a}$$

$$\Sigma_\pm(t) = \left(\mu^2 + \frac{\lambda^2}{2}q\right)q\left(\Delta + G_{\pm 0}(t)\right) + \Sigma_{\pm 0}(t),$$
$$\Sigma_{\pm 0}(t) \equiv \left(\mu^2 + \lambda^2 q\right)S_0(t)G_{\pm 0}(t) + \frac{\lambda^2}{2}S_0^2(t)G_{\pm 0}(t), \tag{113b}$$

where Σ_0, $\Sigma_{\pm 0}$ consist of the terms nonlinear in correlators S_0, $G_{\pm 0}$ and the terms proportional to $S_0\Delta \simeq 0$ are disregarded.

In the ω–representation, inserting the Fourier–transform of Eqs.(110), (113a) in the Dyson equation (108a), and taking into account Eq.(107) we have

$$q_0\left[1 - w\chi_0^2 - \frac{1}{2}\left(\mu^2 + \frac{\lambda^2}{3}q_0\right)q_0\chi_0^2\right] = h^2\chi_0^2, \tag{114}$$

$$S_0 = \frac{(1 + \Sigma_0)G_+G_-}{1 - \left[w + (\mu^2 + \lambda^2 q/2)q\right]G_+G_-}. \tag{115}$$

[2] It is convenient to use the unique response function $G_-(\omega)$ for definition of both susceptibilities χ_0 and χ, taking into account that quantities $\chi_0 \equiv G_-(\omega = 0)$, and $\chi \equiv G_-(\omega \to 0)$ correspond to the equilibrium (macroscopic) and non–equilibrium (microscopic) states. Then Eqs.(65), (89), (90), (96), where the correlators should be labeled by index 0, imply the limit $\omega \to 0$ instead of the exact equality $\omega = 0$.

The first of these equations corresponds to δ–terms ($\omega = 0$) that are caused by the memory effects, whereas the second one — to non–zero frequencies $\omega \neq 0$. In the limit $\omega \to 0$, the product $G_+ G_-$ tends to χ^2, so that the pole of the structure factor (115) determines the point of ergodicity breaking for the thermodynamic system

$$\chi_0^{-2} = w + \left(\mu^2 + \frac{\lambda^2}{2} q_0\right) q_0. \tag{116}$$

Substituting Eq.(113b) into the Dyson equation (108b) yields the relation for retarded Green function

$$G_-^{-1} + \left[(v+w) + \left(\mu^2 + \frac{\lambda^2}{2} q\right) q\right] G_- + \Sigma_{-0} - (1 - \mathrm{i}\omega) = 0 \tag{117}$$

where the ω–representation is used. Then, from Eq.(96c) the equation for the thermodynamic susceptibility $\chi \equiv G_-(\omega \to 0)$ is derived

$$1 - \chi + (v+w)\chi^2 + \frac{\mu^2}{2}\chi\left[(\chi+q)^2 - q^2\right] + \frac{\lambda^2}{6}\chi\left[(\chi+q)^3 - q^3\right] = 0. \tag{118}$$

The macroscopic memory parameter q_0 is given by the equation

$$\left(\frac{\mu^2}{2} + \frac{\lambda^2}{3} q_0\right) q_0^2 = h^2, \tag{119}$$

which is obtained from Eqs.(114), (116) in the limit $\omega = 0$.

8 Discussion

Within the framework of the model under consideration the system of Eqs.(114), (118), (116), (119), (112) provides the complete analytical description of the non–ergodic thermodynamic system with quenched disorder. Eqs.(114) and (118) are similar to the equations obtained by Sherrington and Kirkpatrick for determination of isothermal χ_0 and thermodynamic χ susceptibilities and corresponding memory parameters q_0, q [1]. The equation (116) defines the point T_0 of ergodicity breaking and Eq.(112) gives non-ergodicity parameter Δ. The above consideration implies that one should distinguish the macroscopic quantity q_0, χ_0 and microscopic ones q, χ (the former correspond to frequency $\omega = 0$, the latter — to the limit $\omega \to 0$). The peculiarity of such a hierarchy is that macroscopic values q_0, χ_0 depend on the amplitude h of quenched disorder only, whereas microscopic ones q, χ — on temperature T. Respectively, Eqs. (114), (116), where the temperature T should be taken equal to its value on the ergodicity breaking curve $T_0(h)$, give the macroscopic values q_0, χ_0. What about determination of the microscopic ones q, χ, the Eq.(118) must be added by the equation type of Eq.(114)

$$q\left[1 - w\chi_0^2 - \frac{1}{2}\left(\mu^2 + \frac{\lambda^2}{3}q\right)q\chi_0^2\right] = h^2\chi_0^2, \tag{120}$$

where the memory parameter q is taken as microscopic in character.

It is well to bear in mind that the field h, anharmonicity parameters λ, μ, and interaction parameter w, as well as the inverse values of susceptibilities χ_0, χ have been measured in units of temperature T. Further, it is convenient to choose the following measure units:

$$T_s = \left(\frac{3}{2}\right)^{3/2}\frac{\mu^4}{\lambda^3},\ h_s = \frac{3}{2}\frac{\mu^3}{\lambda^2},\ v_s = w_s = \left(\frac{3}{2}\right)^{-1/2}\lambda,$$
$$\chi_s = \left(\frac{3}{2}\right)^{-1/2}\frac{\lambda}{\mu^2},\ q_s = \frac{3}{2}\frac{\mu^2}{\lambda^2} \equiv u \tag{121}$$

for quantities T, h, v, w, χ, q respectively. As a result, the key equations take the final form:

$$(1 - uT\chi) + (v + w)T\chi^2 + (\chi/2T)\left[(T\chi + q)^2 - q^2\right] + (\chi/4T)\left[(T\chi + q)^3 - q^3\right] = 0, \tag{122a}$$

$$wT + (1 + q/2)\,q/2 + h^2/q = \chi_0^{-2}, \tag{122b}$$

$$wT + (1 + q_0/2)\,q_0/2 + h^2/q_0 = \chi_0^{-2}, \tag{122c}$$

$$[1 + (3/4)q_0]\,q_0 + wT_0 = \chi_0^{-2}, \tag{122d}$$

$$(1 + q_0)q_0^2 = 2h^2, \tag{122e}$$

$$\Delta = uT(\chi_0 - \chi). \tag{122f}$$

Behaviour of the system is specified by the last parameter u in Eqs.(121), that determines the relation between cubic and quartic anharmonicities. In the case $u \ll 1$ main contribution is due to the quartic term (81), whereas at $u \gg 1$ the cubic anharmonicity (91) dominates. The former limit corresponds to strong quenched disorder $h \gg h_s$, the latter — to weak field $h \ll h_s$.

According to Eq.(122e), the dependence of the macroscopic memory parameter q_0 on the quenched disorder amplitude h is governed by the ratio between magnitude h and characteristic field h_s. The linear dependence

$$q_0 = 2^{1/2}(h/\mu), \quad u \gg 1, \tag{123}$$

is realized in the limit $h \ll h_s$, whereas the power relation

$$q_0 = 3^{1/3}(h/\lambda)^{2/3}, \quad u \ll 1 \tag{124}$$

corresponds to the case of $h \gg h_s$ (in Eqs.(123), (124) measured units are restored). The dependence $q_0(h)$ is depicted in Fig.1.

For temperatures above the point of ergodicity breaking T_0 the thermodynamic χ, q and adiabatic χ_0, q_0 values of susceptibilities and memory parameters, as well as Eqs.(122b), (122c) coincide, so that Eqs.(122a),(122c) deteremine the dependencies χ and q on temperature. Accounting Eq.(122d) gives the temperature of the ergodicity breaking $T_0(h)$ as a function of field h (see Fig.2). The peculiarity of the dependence $T_0(h)$ is that the ergodicity breaking temperature takes non–zeroth value $T_{00} \equiv T_0(h=0)$ at $h=0$. Below the ergodicity breaking temperature T_0 Eqs.(122a), (122b) give the microscopic values of the memory parameter q and the susceptibility χ that differs from the macroscopic one χ_0 being constant. As a result, we obtaine the typical temperature dependencies of susceptibilities χ, χ_0 as shown in Fig.3. It is seen that, in accordance with Eq.(122a), at $T < T_0$ the thermodynamic susceptibility $\chi \neq 0$ only if temperature is above the freezing point T_f which is determined by the condition $\partial\chi/\partial T = \infty$, leading to the equation

$$(v+w)T_f + T_f\chi + q + (3/4)(T_f\chi + q)^2 = \chi^{-2}, \tag{125}$$

where χ, q are taken at $T = T_f$.

The phase diagrams, which depict the ranges of possible thermodynamic states on the plane $h - T$ for various values of interaction parameters w and v, are shown in Fig.2. In the limit $h = 0$ the temperatures of ergodicity breaking and freezing are as follows:

$$T_{00} = w\left\{\left(1 + \frac{v}{2w} + \frac{1}{12}\frac{\lambda^2}{w^2}\right) + \left[\left(1 + \frac{v}{2w} + \frac{1}{12}\frac{\lambda^2}{w^2}\right)^2 + \frac{1}{2}\frac{\mu^2}{w^2}\right]^{1/2}\right\}^2, \tag{126a}$$

$$T_f \approx 4(v+w)\left(1 + \frac{\mu^2 + (2/3)\lambda^2}{4(v+w)^2}\right), \tag{126b}$$

where measured units are restored and the second equality is for $\mu^2, \lambda^2 \ll w^2$. So, cubic and quartic anharmonicities result in an increase of both ergodicity breaking and freezing temperatures. When quenched disorder is large, $q_0, q_0^2 \gg wT_0$, the isothermal susceptibility χ_0 in Eq.(122d) is small, and Eqs.(122a), (122c), (122d) provide the estimate $\chi_0 \approx 2/uT_0$. Then, for measured quantities one has:

$$T_0 \approx 2^{5/4}\ \mu\ (h/\mu)^{1/2}, \qquad (w/\mu)^2\mu \ll h \ll (\mu/\lambda)^2\mu; \tag{127a}$$

$$T_0 \approx 2^{1/2}3^{1/3}\ \lambda\ (h/\lambda)^{2/3}, \qquad h \gg (\mu/\lambda)^2\mu,\ (w/\lambda)^{3/2}\ \lambda \tag{127b}$$

for $\mu^2 \gg \lambda^2$ and $\lambda^2 \gg \mu^2$, respectively. So, the non–ergodicity domain is extended indefinitely at strong increasing of quenched disorder. As it is shown in Figs.2a,b, and Fig.4a,b, the dependencies $T_0(h)$, $T_f(h)$ is non–monotonous if either $w < 0.5$ or $v > 1$.

Influence of the interaction parameters w, v and the anharmonicity ratio u on the temperature dependence of the susceptibility χ is illustrated in Fig.5. According to Fig.5a, increasing w causes a decrease of χ and an increase of temperatures T_0, T_f. The same behaviour is revealed at increasing parameter v (Fig.5b). By contrast, tendency is opposite under an increase in u (see Fig.5c).

Finally, let us consider behaviour of the non–ergodicity Δ and memory q parameters that are determined by complete system of equations (122). Corresponding dependencies on temperature are shown in Fig.6a,b. At the freezing state, where $\chi \equiv 0$, the non–ergodicity parameter (122f) linearly depends on temperature because the isothermal susceptibility χ_0 is constant. The appearance of finite value of the thermodynamic susceptibility χ above the freezing point T_f results in step–like decrease of the value Δ. With further growth of temperature the irreversible response $\Delta(T)$ monotonously decays taking zero value at the ergodicity breaking point T_0 (see Fig.6a). With increasing temperature from 0 to T_0 microscopic memory parameter q monotonously decriases, taking minimal value at the ergodicity breaking point T_0. Above this point $q(T)$ increases (see Fig.6b). It is seen, that the quenched disorder encrease extendes the temperature domain of the non–ergodicity and causes growth of the memory parameter. In a spirit of generalized picture of phase transition it can be attributed to the fact that the microscopic memory parameter q above the point T_0 corresponds to a soft mode that transforms to a mode of ergodicity restoring below the temperature T_0. The non–ergodicity parameter Δ represents the order parameter.

The analytical expressions for dependencies $\Delta(T)$, $q(T)$ can be obtained only near the ergodicity breaking curve $T_0(h)$. For $h = 0$ and $T_0 = T_{00}$, from Eq.(122a) assuming that $0 < T_{00} - T \ll T_{00}$, $\chi \approx \chi_{00} - \Delta/uT_{00}$, $\Delta \ll u\chi_{00}T_{00}$, up to the first order in small parameters $\varepsilon \equiv T/T_{00} - 1$ and $\Delta(u\chi_{00}T_{00})^{-1}$ we have for measured units

$$\Delta = -A_0\varepsilon, \quad A_0 \equiv \frac{T_{00}}{w}\left(\frac{w}{\mu}\right)^2 \frac{1 - \frac{\lambda^2}{6w^2}}{1 + \left(\frac{\lambda^2}{2\mu^2} + \frac{vw}{\mu^2}\right)\left(\frac{T_{00}}{w}\right)^{1/2}}, \quad \varepsilon < 0; \tag{128a}$$

$$q = Q\varepsilon, \quad Q \equiv \frac{4}{3}\frac{T_{00}}{w}\left(\frac{\lambda w}{\mu^2}\right)^2 \frac{1 - \frac{\lambda^2}{12w^2}}{1 + \frac{\lambda^2}{2\mu^2}\left(\frac{T_{00}}{w}\right)^{1/2}}, \quad \varepsilon > 0, \tag{128b}$$

In the case of $h \neq 0$ the result for temperature dependence is

$$\Delta = -A\varepsilon, \quad A \equiv \frac{2}{\lambda^2\chi_0^2}\left(\frac{1 - \frac{w}{2}\chi_0^2 T_0 - \frac{\lambda^2}{12}\chi_0^4 T_0^2}{\frac{v}{\lambda^2\chi_0} + \frac{\mu^2}{\lambda^2} + q_0 + \frac{1}{2}\chi_0 T_0}\right), \quad \Delta,\ \varepsilon < 0. \tag{129}$$

Correspondingly, at the fixed temperature Eq.(122a) gives in the linear approximation $0 < q_0 - q \ll q_0$:

$$\Delta = B\,(q - q_0)\,, \quad B^{-1} \equiv 1 + \frac{v}{\lambda^2\chi_0}\left(\frac{\mu^2}{\lambda^2} + q_0 + \frac{1}{2}\chi_0 T_0\right)^{-1}. \tag{130}$$

A

For nilpotent representation let us rewrite the Lagrangian (10) in the form of Euclidean field theory [8]

$$L = \kappa + \pi, \tag{A.1}$$

where the kinetic κ and potential π energies are

$$\kappa = \varphi\dot{\eta} - \varphi^2/2, \tag{A.2}$$
$$\pi = (\partial V/\partial\eta)\varphi. \tag{A.3}$$

In order to obtain the nilpotent form (13a) of the kinetic energy (A.2), we have to determine the operator D. The complete form of the dependence of the operator D on the nilpotent coordinate ϑ is presented by the expression

$$D = a + b(\partial/\partial\vartheta) + c\vartheta + d\vartheta(\partial/\partial\vartheta), \tag{A.4}$$

where the coefficients a, b, c, d are unknown operators. The substitution of Eqs.(11), (A.4) into Eq.(13a) and taking into account the properties (12) leads to the expression (A.2) with the following coefficients:

$$a = \partial_t, \qquad b = -1, \qquad c = 0, \qquad d = -2\partial_t, \tag{A.5}$$

where $\partial_t \equiv \partial/\partial t$ is the derivative with respect to time. As a result, the operator (A.4) takes the form (14). It has the property

$$D^2 = \partial_t^2. \tag{A.6}$$

While considering the definitions (11), (12), (14) it is easy to see that D is a Hermite operator.

Under the infinitesimal transformation $\delta \equiv e^{\varepsilon D} - 1 \simeq \varepsilon D$ with the parameter $\varepsilon \to 0$, the values t and ϑ acquire the additions $\delta t = \varepsilon$, $\delta\vartheta = -\varepsilon$ which differ in the sign. Considering the corresponding field addition $|\delta\phi_\varphi| = \varepsilon|D_\varphi||\phi_\varphi|$, it is convenient to use the matrix form for the nilpotent field (11) and the operator D:

$$|\phi_\varphi| = \begin{pmatrix}\eta\\ \varphi\end{pmatrix}, \qquad |D_\varphi| = \begin{pmatrix}\partial_t & -1\\ 0 & -\partial_t\end{pmatrix}, \qquad \partial_t \equiv \frac{\partial}{\partial t}. \tag{A.7}$$

According to Eq.(A.7), the change of the order parameter is proportional to a difference between the rate of change of the order parameter and the fluctuation amplitude, whereas the change of the latter is proportional to its rate with the opposite sign.

To prove the equivalence of the term (A.3) in the Lagrangian (A.1) and the Grassmann potential energy (13b), let us carry out the formal expansion of the thermodynamic potential in powers of the component $\vartheta\varphi$ of Eq.(11):

$$\pi = \int \left[V(\eta) + \frac{\delta V}{\delta \eta}\ \varphi\vartheta \right] \mathrm{d}\vartheta. \tag{A.8}$$

Here all the terms of powers higher than 1 are omitted according to the nilpotent condition. Using the integration properties (12), we obtain immediately Eq.(A.3) as it was required.

In the case of the two–component nilpotent field (27), the consideration is fulfilled by analogy. For brevity, let us point out the difference between (27) and the above–considered case (11) only. The corresponding infinitesimal transformation $\delta \simeq \varepsilon D$, where the generator D is given by Eq.(28), results in the additions $\delta t = 0$, $\delta\vartheta = -\varepsilon$, $|\delta\phi_f| = \varepsilon |D_f||\phi_f|$ in which the matrix form

$$|\phi_f| = \begin{pmatrix} \eta \\ -f \end{pmatrix}, \qquad |D_f| = \begin{pmatrix} 0 & -1 \\ -\partial_t^2 & 0 \end{pmatrix}, \qquad \partial_t \equiv \frac{\partial}{\partial t} \tag{A.9}$$

is used. The generator D_f has the property (A.6). It is easy seen that the Lagrangians (10), (23) are invariant under the transformations given by the generators D_φ, D_f, respectively, provided that the infinitesimal parameter ε is pure imaginary, and the fields $\eta(\mathbf{r}, t)$, $\varphi(\mathbf{r}, t)$, $f(\mathbf{r}, t)$ are complex–valued. So, for real fields the two–component representations (11), (27) are just convenient approximations.

The matrices of the transformation between the fields (A.9) and (A.7) (see Eqs.(29), (32)) take the form

$$|\tau_\pm| = \begin{pmatrix} 1 & 0 \\ \pm\partial_t & 1 \end{pmatrix}. \tag{A.10}$$

Let us consider the four–component SUSY fields (35), (44). Instead of Eq.(A.6), the corresponding couples of operators (37), (45) satisfy the conditions:

$$\begin{aligned} &\mathcal{D}^2 = \overline{\mathcal{D}}^2 = 0, \quad \{\overline{\mathcal{D}}, \mathcal{D}\} = -2\partial_t, \quad [\overline{\mathcal{D}}, \mathcal{D}]^2 = (2\partial_t)^2; \\ &\{\mathcal{D}_\varphi, \mathcal{D}_f\} = \{\overline{\mathcal{D}}_\varphi, \overline{\mathcal{D}}_f\} = 0, \quad \{\overline{\mathcal{D}}_\varphi, \mathcal{D}_f\} = -\partial_t, \quad \{\overline{\mathcal{D}}_f, \mathcal{D}_\varphi\} = -3\partial_t, \end{aligned} \tag{A.11}$$

where the curly and square brackets denote anticommutator and commutator, respectively. The generalized anticommutation rules for the operators $\mathcal{D}^{(\pm)} \equiv \mathcal{D}(\pm t)$, $\overline{\mathcal{D}}^{(\pm)} \equiv \overline{\mathcal{D}}(\pm t)$ corresponding to the opposite directions of the time t, read:

$$\{\mathcal{D}^{(\pm)}, \mathcal{D}^{(\mp)}\} = \{\overline{\mathcal{D}}^{(\pm)}, \overline{\mathcal{D}}^{(\mp)}\} = \{\overline{\mathcal{D}}_f^{(\pm)}, \mathcal{D}_f^{(\mp)}\} = 0,$$

$$\{\overline{\mathcal{D}}^{(\pm)},\mathcal{D}^{(\pm)}\}=-\{\overline{\mathcal{D}}^{(\pm)}_{\varphi},\mathcal{D}^{(\mp)}_{\varphi}\}=\mp 2\partial_t;$$
$$\{\mathcal{D}^{(\pm)}_{\varphi},\mathcal{D}^{(\pm)}_{f}\}=\{\overline{\mathcal{D}}^{(\pm)}_{\varphi},\overline{\mathcal{D}}^{(\pm)}_{f}\}=0,\quad \{\overline{\mathcal{D}}^{(\pm)}_{\varphi},\mathcal{D}^{(\pm)}_{f}\}=\mp\partial_t,\quad \{\overline{\mathcal{D}}^{(\pm)}_{f},\mathcal{D}^{(\pm)}_{\varphi}\}=\mp 3\partial_t;$$

$$\{\mathcal{D}^{(\pm)}_{\varphi},\mathcal{D}^{(\mp)}_{f}\}=\{\overline{\mathcal{D}}^{(\pm)}_{\varphi},\overline{\mathcal{D}}^{(\mp)}_{f}\}=0,\qquad \{\overline{\mathcal{D}}^{(\pm)}_{\varphi},\mathcal{D}^{(\mp)}_{f}\}=\{\overline{\mathcal{D}}^{(\pm)}_{f},\mathcal{D}^{(\mp)}_{\varphi}\}=\pm\partial_t. \tag{A.12}$$

In Eqs.(A.11), (A.12) the coincident indexes are suppressed. The simplest way to prove Eqs.(A.11), (A.12) is to introduce the four–rank matrices (see Eqs.(A.7), (A.9)):

$$|\Phi_{\varphi}|=\begin{pmatrix}\eta\\ \psi\\ -\overline{\psi}\\ \varphi\end{pmatrix},\quad |\mathcal{D}_{\varphi}|=\begin{pmatrix}0 & 1 & 0\,0\\ 0 & 0 & 0\,0\\ -2\partial_t & 0 & 0\,1\\ 0 & 2\partial_t & 0\,0\end{pmatrix},\quad |\overline{\mathcal{D}}_{\varphi}|=\begin{pmatrix}0\,0\,1 & 0\\ 0\,0\,0 & -1\\ 0\,0\,0 & 0\\ 0\,0\,0 & 0\end{pmatrix}; \tag{A.13}$$

$$|\Phi_{f}|=\begin{pmatrix}\eta\\ \psi\\ -\overline{\psi}\\ -f\end{pmatrix},\quad |\mathcal{D}_{f}|=\begin{pmatrix}0 & 1\,0\,0\\ 0 & 0\,0\,0\\ -\partial_t & 0\,0\,1\\ 0 & \partial_t\,0\,0\end{pmatrix},\quad |\overline{\mathcal{D}}_{f}|=\begin{pmatrix}0 & 0 & 1 & 0\\ -\partial_t & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -\partial_t & 0\end{pmatrix}. \tag{A.14}$$

The matrices of the transformation between the fields (A.14) and (A.13) take the form (cf. Eq.(A.10))

$$|T_{\pm}|=\begin{pmatrix}1 & 0\,0\,0\\ 0 & 1\,0\,0\\ 0 & 0\,1\,0\\ \pm\partial_t & 0\,0\,1\end{pmatrix}. \tag{A.15}$$

The infinitesimal transformations $\delta\simeq\overline{\varepsilon}\mathcal{D}$, $\overline{\delta}\simeq\overline{\mathcal{D}}\varepsilon$ give the following additions:

$$\delta_{\varphi}\theta=0,\quad \delta_{\varphi}\overline{\theta}=\overline{\varepsilon},\quad \delta_{\varphi}t=-2\overline{\varepsilon}\theta,\quad \overline{\delta}_{\varphi}\theta=\varepsilon,\quad \overline{\delta}_{\varphi}\overline{\theta}=0,\quad \overline{\delta}_{\varphi}t=0;$$
$$\delta_{f}\theta=0,\quad \delta_{f}\overline{\theta}=\overline{\varepsilon},\quad \delta_{f}t=-\overline{\varepsilon}\theta,\quad \overline{\delta}_{f}\theta=\varepsilon,\quad \overline{\delta}_{f}\overline{\theta}=0,\quad \overline{\delta}_{f}t=-\overline{\theta}\varepsilon; \tag{A.16}$$

$$|\delta\Phi_{\varphi}|=\overline{\varepsilon}|\mathcal{D}_{\varphi}||\Phi_{\varphi}|,\quad |\overline{\delta}\Phi_{\varphi}|=|\overline{\mathcal{D}}_{\varphi}||\Phi_{\varphi}|\varepsilon;$$
$$|\delta\Phi_{f}|=\overline{\varepsilon}|\mathcal{D}_{f}||\Phi_{f}|,\quad |\overline{\delta}\Phi_{f}|=|\overline{\mathcal{D}}_{f}||\Phi_{f}|\varepsilon.$$

At last, the equation for the SUSY field (35)

$$\int V(\Phi_\varphi)\mathrm{d}^2\theta = \frac{\delta V}{\delta \eta}\ \varphi - \overline{\psi}\ \frac{\delta^2 V}{\delta \eta^2}\ \psi \tag{A.17}$$

is obtained by analogy with Eq.(A.8) to represent the terms in Eq.(34) that contain the potential $V\{\eta\}$ in SUSY form. In the case of the field (44), the multiplier φ must be substituted by $-f$.

B

Following the standard field scheme [4], let us show how the four-component SUSY field (44) is split into a couple of chiral two–component Grassmann conjugated fields $\Phi_\pm$. These SUSY fields are obtained from the initial SUSY field Φ_f under the following transformations:

$$\Phi_\pm = T_\pm \Phi_f; \quad T_\pm \equiv e^{\pm\partial}, \quad \partial \equiv \overline{\theta}\theta\partial_t, \qquad \partial_t \equiv \partial/\partial t. \tag{B.1}$$

Accordingly, the generators (45) take the form

$$\mathcal{D}_\pm = T_\pm \mathcal{D}_f T_\mp \qquad \overline{\mathcal{D}}_\pm = T_\pm \overline{\mathcal{D}}_f T_\mp. \tag{B.2}$$

Due to the Grassmann nature of the parameter ∂ in the operators $T_\pm$, it is convenient to rewrite (B.2) in the following form:

$$\mathcal{D}_\pm = \mathcal{D}_f \pm [\partial, \mathcal{D}_f], \qquad \overline{\mathcal{D}}_\pm = \overline{\mathcal{D}}_f \pm \left[\partial, \overline{\mathcal{D}}_f\right], \tag{B.3}$$

where the square brackets denote commutator. In explicit form one has

$$\begin{aligned} &\mathcal{D}_+ = \partial/\partial\overline{\theta} - 2\theta\partial_t, \qquad \mathcal{D}_- = \partial/\partial\overline{\theta}; \\ &\overline{\mathcal{D}}_+ = \partial/\partial\theta, \qquad \overline{\mathcal{D}}_- = \partial/\partial\theta - 2\overline{\theta}\partial_t. \end{aligned} \tag{B.4}$$

Apparently, the operators $\mathcal{D}_+$, $\overline{\mathcal{D}}_+$ coincide with the generators $\mathcal{D}_\varphi$, $\overline{\mathcal{D}}_\varphi$, Eqs.(37).

According to Eqs.(44), the definitions (B.1) give

$$\Phi_\pm = \eta + \overline{\theta}\psi + \overline{\psi}\theta \pm \overline{\theta}\theta\,(\dot{\eta} \mp f), \tag{B.5}$$

where the point denotes the derivative with respect to time. The comparison of Eq.(B.5) with the definition (35) gives the identity $\Phi_+ \equiv \Phi_\varphi$. The action of the operators (B.4) on Eq.(B.5) gets

$$\mathcal{D}_{\pm}\Phi_{\pm} = \psi - \theta\,(\dot{\eta} + f) + \underline{2\overline{\theta}\theta\dot{\psi}},$$
$$-\overline{\mathcal{D}}_{\mp}\Phi_{\mp} = \overline{\psi} + \overline{\theta}\,(\dot{\eta} - f) - \underline{2\overline{\theta}\theta\dot{\overline{\psi}}}, \tag{B.6}$$

where the underlined terms concern only the upper indexes of the left–hand parts.

The chiral SUSY fields are fixed by the gauge conditions [4]

$$\mathcal{D}_{-}\Phi_{-} = 0, \qquad \overline{\mathcal{D}}_{+}\Phi_{+} = 0, \tag{B.7}$$

which, in accordance with the definitions (B.4) signify, that Φ_{-} and Φ_{+} are independent of $\overline{\theta}$ and θ, respectively. On the other hand, taking into account Eqs.(B.6), the gauge (B.7) results in the equations

$$\psi - \theta\,(f + \dot{\eta}) = 0,$$
$$\overline{\psi} + \overline{\theta}\,(\dot{\eta} - f) = 0 \tag{B.8}$$

for Φ_{-} and Φ_{+}, correspondingly. Substituting Eqs.(B.8) into Eq.(B.5), the final expressions for the chiral SUSY fields are obtained:

$$\phi_{-} = \eta + \overline{\psi}\theta,$$
$$\phi_{+} = \eta + \overline{\theta}\psi. \tag{B.9}$$

These equations give the non–reducible representations of the SUSY fields (35), (44) under the conditions of the gauge (B.7). The chiral field $\phi_{+}(t)$ corresponds to the positive direction of the time t, whereas $\phi_{-}(t)$ is related to the negative one [4].

C

Let us consider the invariance properties of the SUSY action

$$S = \int \Big[K(\Phi(z)) + V(\Phi(z))\Big]\mathrm{d}z, \quad K(\Phi) \equiv \frac{1}{2}(\overline{\mathcal{D}}\Phi)(\mathcal{D}\Phi), \quad z \equiv \{\mathbf{r}, t, \theta, \overline{\theta}\} \tag{C.1}$$

under the Grassmann conjugated transformations

$$\delta\Phi = \sum_{\alpha}\overline{\varepsilon}_{\alpha}\mathcal{D}^{(\alpha)}\Phi, \qquad \overline{\delta}\Phi = \sum_{\alpha}\overline{\mathcal{D}}^{(\alpha)}\Phi\varepsilon_{\alpha}, \tag{C.2}$$

given by the SUSY generators $\mathcal{D}^{(\alpha)}$, $\overline{\mathcal{D}}^{(\alpha)}$ which differ in the time t and Grassmann coordinates θ, $\overline{\theta}$. According to Eq.(A.17), the potential term in Eq.(C.1) is SUSY invariant if the kernel $V(\eta)$ does not depend on the time t. Up to inessential total time derivatives, the Grassmann conjugated variations of the remaining kinetic term

$$\delta K = \frac{1}{2}\overline{\mathcal{D}}\left(\sum_\alpha \overline{\varepsilon}_\alpha \mathcal{D}^{(\alpha)}\Phi\right)(\mathcal{D}\Phi) + \frac{1}{2}(\overline{\mathcal{D}}\Phi)\mathcal{D}\left(\sum_\alpha \overline{\varepsilon}_\alpha \mathcal{D}^{(\alpha)}\Phi\right), \tag{C.3}$$

$$\overline{\delta} K = \frac{1}{2}\overline{\mathcal{D}}\left(\sum_\alpha \overline{\mathcal{D}}^{(\alpha)}\Phi\varepsilon_\alpha\right)(\mathcal{D}\Phi) + \frac{1}{2}(\overline{\mathcal{D}}\Phi)\mathcal{D}\left(\sum_\alpha \overline{\mathcal{D}}^{(\alpha)}\Phi\varepsilon_\alpha\right)$$

can be rewritten in the form

$$\delta K = \frac{1}{2}\sum_\alpha \overline{\varepsilon}_\alpha \mathcal{D}^{(\alpha)}\left[(\overline{\mathcal{D}}\Phi)(\mathcal{D}\Phi)\right], \qquad \overline{\delta} K = \frac{1}{2}\sum_\alpha \overline{\mathcal{D}}^{(\alpha)}\left[(\overline{\mathcal{D}}\Phi)(\mathcal{D}\Phi)\right]\varepsilon_\alpha \tag{C.4}$$

provided that the anticommutators $\{\mathcal{D}, \mathcal{D}^{(\alpha)}\}$, $\{\overline{\mathcal{D}}, \mathcal{D}^{(\alpha)}\}$, $\{\mathcal{D}, \overline{\mathcal{D}}^{(\alpha)}\}$, $\{\overline{\mathcal{D}}, \overline{\mathcal{D}}^{(\alpha)}\}$ are either equal to zero or proportional to the derivative with respect to the time ∂_t. According to Eqs.(A.11), (A.12) such conditions are fulfilled if only the SUSY generators $\mathcal{D}^{(\alpha)}$, $\overline{\mathcal{D}}^{(\alpha)}$ either coincide with the initial operators $\mathcal{D}$, $\overline{\mathcal{D}}$, or are reduced to the transformed operators $\mathcal{D}_\pm$, $\overline{\mathcal{D}}_\pm$ determined by equations of (B.2) type, or correspond to the opposite time directions $\mathcal{D}_\pm^{(\pm)}$, $\overline{\mathcal{D}}_\pm^{(\pm)}$. Being reduced to the derivatives with respect to the time t and Grassmann coordinate θ, $\overline{\theta}$, these operators inserted into Eqs.(C.4) give, as it was required, zero for the variations of the corresponding action (C.1).

Among the above–mentioned generators, the following ones

$$\mathcal{D}_-^{(-)} = \frac{\partial}{\partial\overline{\theta}}, \qquad \overline{\mathcal{D}}_-^{(-)} = \frac{\partial}{\partial\theta} + 2\overline{\theta}\frac{\partial}{\partial t} \tag{C.5}$$

and their anticommutator $\{\overline{\mathcal{D}}_-^{(-)}, \mathcal{D}_-^{(-)}\} = 2\partial_t$ (see Eqs.(A.12)) are of a special interest for us. The operators (C.5) are a result of the double action of the transformation T_- on the initial generators $\mathcal{D}_\varphi$, $\overline{\mathcal{D}}_\varphi$, Eqs.(37), that gives the generators $\mathcal{D}_-$, $\overline{\mathcal{D}}_-$, Eqs.(B.3), corresponding to the opposite time directions. Therefore, the generators $\mathcal{D}_-^{(-)} \equiv \mathcal{D}_-(-t)$, $\overline{\mathcal{D}}_-^{(-)} \equiv \overline{\mathcal{D}}_-(-t)$ given by Eqs.(C.5) are related to the initial ones $\mathcal{D}_\varphi(t)$, $\overline{\mathcal{D}}_\varphi(t)$ and play a significant role hereinafter.

Due to the standard manner, it is easy to show that the above conditions $\delta S = 0$, $\overline{\delta} S = 0$ give rise to the Ward identities [8]

$$\sum_{i=1}^n \mathcal{D}_i^{(\alpha)}\Gamma^{(n)}(\{z_i\}) = 0, \qquad \sum_{i=1}^n \overline{\mathcal{D}}_i^{(\alpha)}\Gamma^{(n)}(\{z_i\}) = 0 \tag{C.6}$$

for SUSY n–point $\Gamma^{(n)}$ proper vertices type of the 2–point supercorrelator $C(z_2, z_1)$, Eq.(53) and the self–energy superfunction $\Sigma(z_2, z_1)$. Obviously, under $\mathcal{D}^{(\alpha)} \equiv \partial_t$, $\mathcal{D}^{(\alpha)} \equiv \mathcal{D}_-^{(-)}$ conditions (C.6) mean that above–mentioned supercorrelator depends on $t_2 - t_1$ and $\overline{\theta}_2 - \overline{\theta}_1$ differences only:

$$C_\varphi(z_2, z_1) = S(t_2 - t_1) + (\overline{\theta}_2 - \overline{\theta}_1)\Big[G_+(t_2 - t_1)\theta_2 - G_-(t_2 - t_1)\theta_1\Big] \tag{C.7}$$

where the space dependence is suppressed, for brevity. In accordance with the SUSY field definition (35), one has:

$$
\begin{aligned}
&S(t_2-t_1)=\langle\eta(t_2)\eta(t_1)\rangle\,,\\
&G_+(t_2-t_1)=\langle\varphi(t_2)\eta(t_1)\rangle\,\vartheta(t_1-t_2)=\left\langle\overline{\psi}(t_2)\psi(t_1)\right\rangle\vartheta(t_1-t_2),\\
&G_-(t_2-t_1)=\langle\eta(t_2)\varphi(t_1)\rangle\,\vartheta(t_2-t_1)=\left\langle\overline{\psi}(t_1)\psi(t_2)\right\rangle\vartheta(t_2-t_1)
\end{aligned}
\tag{C.8}
$$

where the step function $\vartheta(t)=1$ for $t>0$ and $\vartheta(t)=0$ for $t<0$. By virtue of the causality principle, the advanced Green function $G_+(t_2-t_1)$ which is the factor before $\overline{\theta}_1\theta_2$, vanishes at $t_1<t_2$, as required. On the other hand, the retarded Green function $G_-(t_2-t_1)=0$ at $t_1>t_2$, to be the coefficient of $\overline{\theta}_2\theta_1$. Moreover, the symmetry condition $C(z_2,z_1)=C(z_1,z_2)$ gives rise to the equations $S(t_2-t_1)=S(t_1-t_2)$, $G_-(t_2-t_1)=G_+(t_1-t_2)$. Inserting the operator $\overline{\mathcal{D}}_-^{(-)}$ into the Ward identity (C.6) results in the equation

$$
2\dot{S}(t)=G_+(t)-G_-(t) \tag{C.9}
$$

that is the fluctuation–dissipation relation. All the above statements hold for the self–energy function $\Sigma(z_2,z_1)$ with the components $\Sigma(t_2-t_1)$, $\Sigma_\pm(t_2-t_1)$ which replace $S(t_2-t_1)$, $G_\pm(t_2-t_1)$, correspondingly.

It is worthwhile to point out specially the relations in Eqs.(C.8) which connect the Bose correlators for the components η, φ, and the Fermi ones for components ψ, $\overline{\psi}$. The above–used Ward identities (C.6) allow to obtain these relations as a trivial consequence of the SUSY field definition (35). But such equations can be obtained also more simply. Indeed, the Fermi correlator $\left\langle\overline{\psi}\psi\right\rangle$ is equal to $\langle(\delta^2V/\delta\eta^2)^{-1}\rangle$ in accordance with Eqs.(21), (34). On the other hand, using the susceptibility definition and Eqs.(22), (26) we have $\langle\eta\varphi\rangle=\langle\delta\eta/\delta\varphi\rangle=\langle(\delta\varphi/\delta\eta)^{-1}\rangle=\langle(\delta^2V/\delta\eta^2)^{-1}\rangle$ for the Bose correlator Q.E.D.

Obviously, to pass to the correlator of the two–component field (11), it is necessary to replace the factors $(\overline{\theta}_2-\overline{\theta}_1)\theta_2$, $(\overline{\theta}_2-\overline{\theta}_1)\theta_1$ in Eq.(C.7) by the nilpotent coordinates ϑ_2, $-\vartheta_1$, respectively, and to omit the term with $\overline{\vartheta}_2\vartheta_1$. The SUSY correlator $C_f(z_2,z_1)=T_-(z_2)T_-(z_1)C_\varphi(z_2,z_1)$ corresponding to the superfields (27) and (44), takes the form

$$
C_f(z_2,z_1)=S+\overline{\theta}_2\theta_2m_+ +\overline{\theta}_1\theta_1m_- -\overline{\theta}_2\theta_1G_- -\overline{\theta}_1\theta_2G_+, \tag{C.10}
$$

where the arguments t_2-t_1 are suppressed in the factors S, $m_\pm$, $G_\pm$. Moreover, in view of Eq.(22), new functions are introduced (cf. Eqs.(C.8))

$$
m_+(t)=\langle\eta(-t)\rangle\,\vartheta(-t)f_{\rm ext},\qquad m_-(t)=\langle\eta(t)\rangle\,\vartheta(t)f_{\rm ext},\qquad f_{\rm ext}\equiv -f \tag{C.11}
$$

to represent the connection between the averaged order parameter $\langle \eta(t) \rangle$ and the external force $f_{\text{ext}} \equiv -f$ (note that the latter is switched at time $t = 0$ and remains constant). The correlators (C.11) are related to the Green functions as follows:

$$G_{\pm}(t) = m_{\pm}(t) \pm \dot{S}(t) \tag{C.12}$$

and possess the symmetry condition $m_{+}(-t) = m_{-}(t)$.

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Captures

Fig.1 Dependence of the macroscopic memory parameter q_0 on the quenched disorder intensity h.

Fig.2 Dependencies of the ergodicity breaking temperature T_0 (solid line), and the freezing temperature T_f (thin line) on the quenched disorder intensity h ($u = 0.5$, $v = 0$) for different values of the effective interaction parameter: a) w=0.5; b) w=0.2.

Fig.3 Temperature dependencies of the thermodynamic and adiabatic susceptibilities χ and χ_0 for: a) different values of the quenched disorder intensity h (curves 1, 2 correspond to $h = 0, 4$) at $w = 0.5$, $u = 0.5$, $v = 0$; b) different values of the effective interaction parameter (curves 1, 2 correspond to $w = 0.5, 0.2$) at $h = 4$, $u = 0.5$, $v = 0$.

Fig.4 Dependencies of the ergodicity breaking temperature T_0 on the quenched disorder intensity h and: a) effective interaction parameter w at $u = 0.5$, $v = 0$; b) proper interaction parameter v at $u = 1$, $w = 0.5$.

Fig.5 The shift of the temperature dependencies of the thermodynamic susceptibility $\chi(T)$ caused by variation of: a) effective interaction parameter w at $u = 0.5$, $v = 0$ (curves 1, 2, 3 correspond to w =0.5, 1, 1.5); b) proper interaction parameter v at $u = 0.5$, $w = 0.5$ (curves 1, 2, 3 correspond to v=0, 1, 1.5); c) anharmonicity ratio u at $w = 0.5$, $v = 0$ (curves 1, 2, 3 correspond to u =0.5, 1, 1.5).

Fig.6 Temperature dependencies ($u = w = 0.5$, $v = 0$) of: a) non–ergodicity parameter Δ; b) microscopic memory parameter q. (curves 1, 2 correspond to h =0, 4).

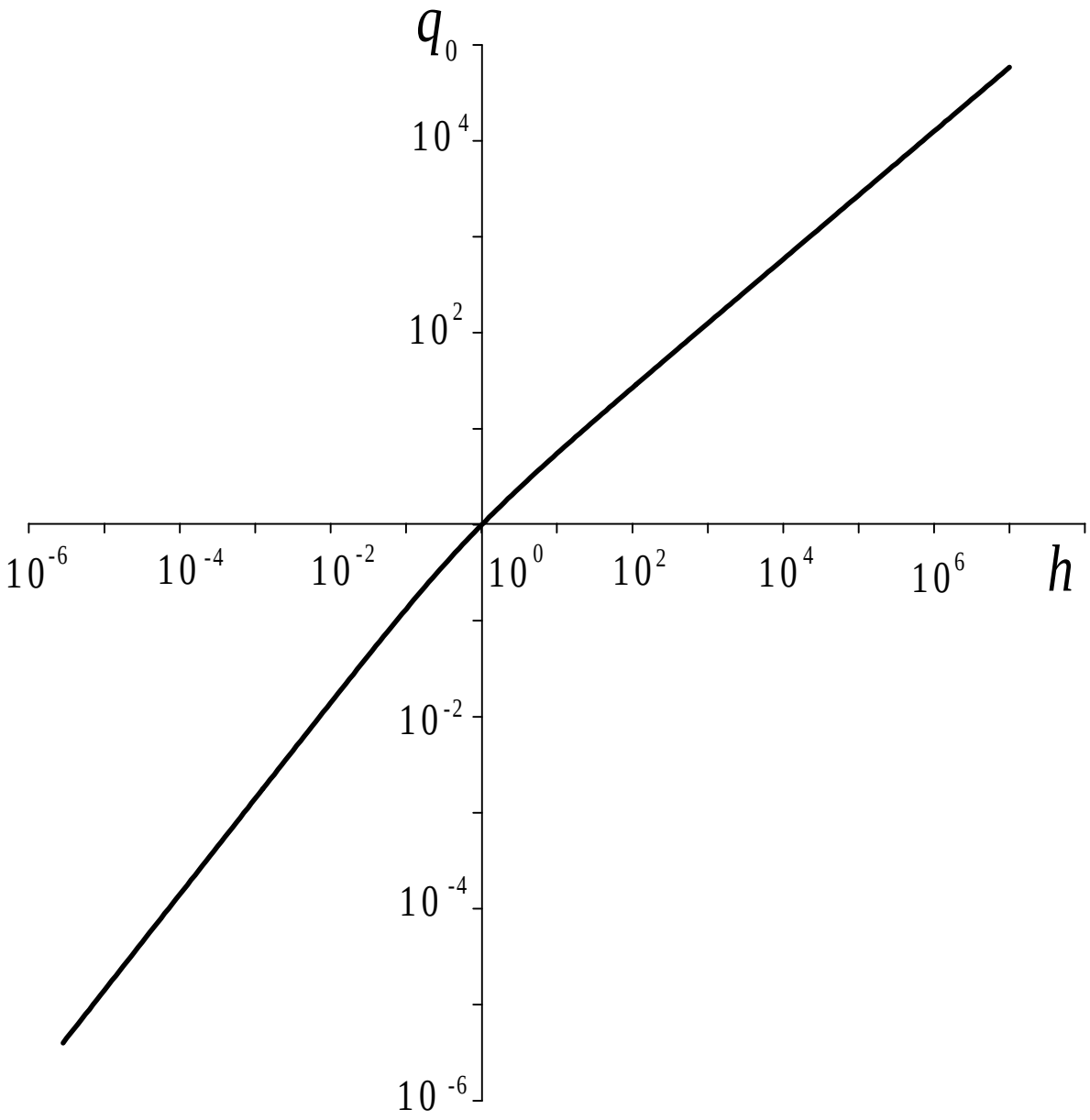
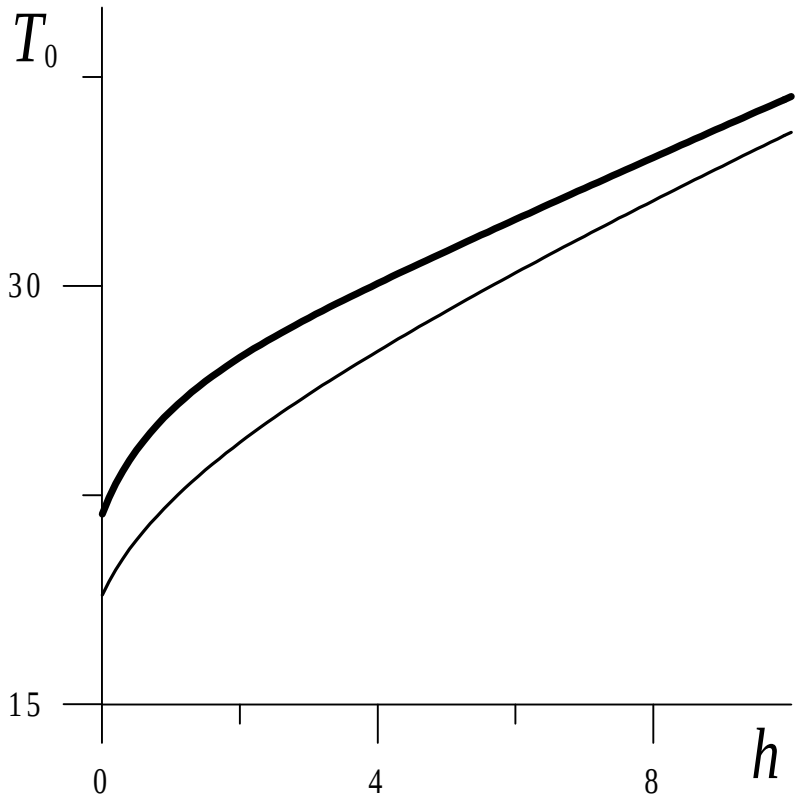


Fig.1

a



b

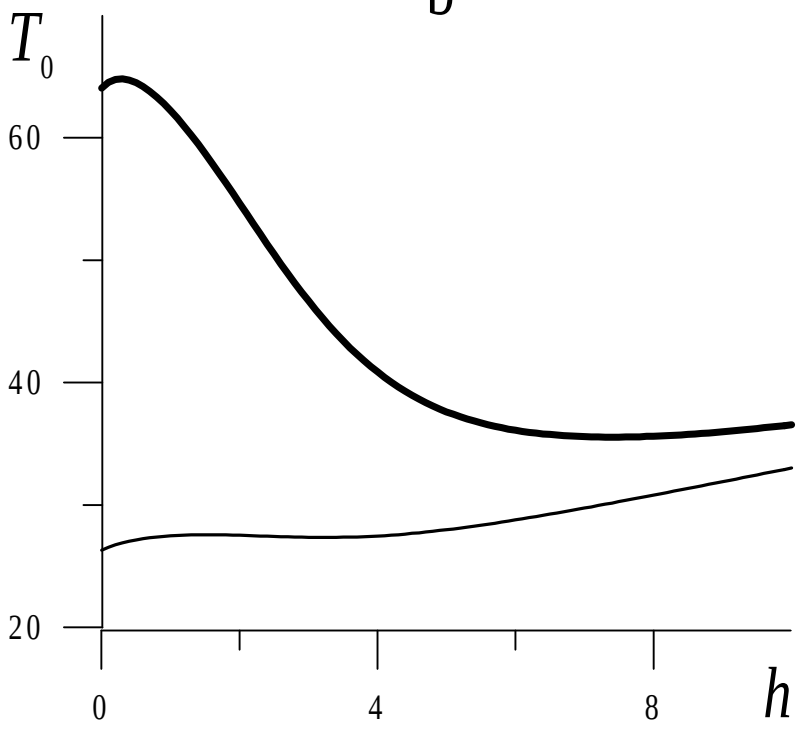
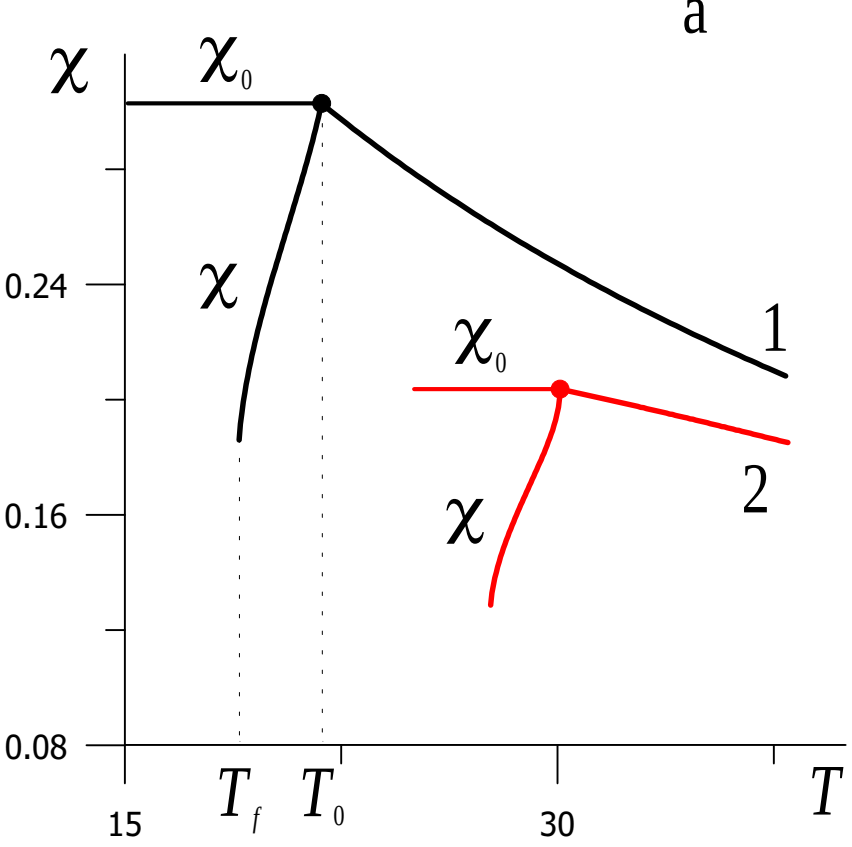


Fig.2


a
χ
χ0
χ
0.24
0.16
0.08
χ0
1
2
χ
15
Tf
T0
30
T

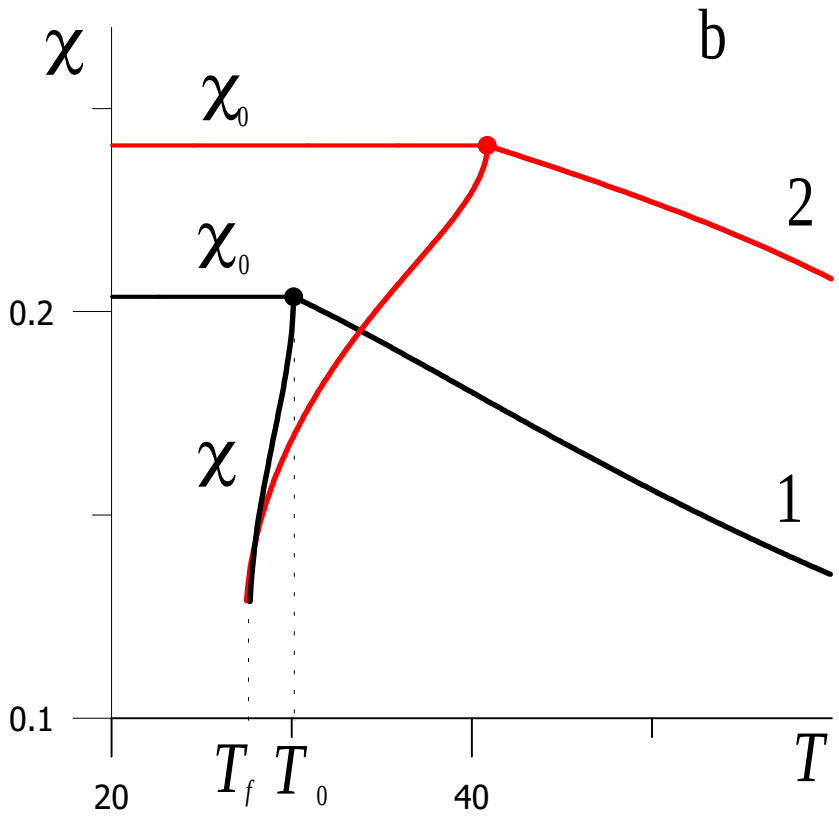
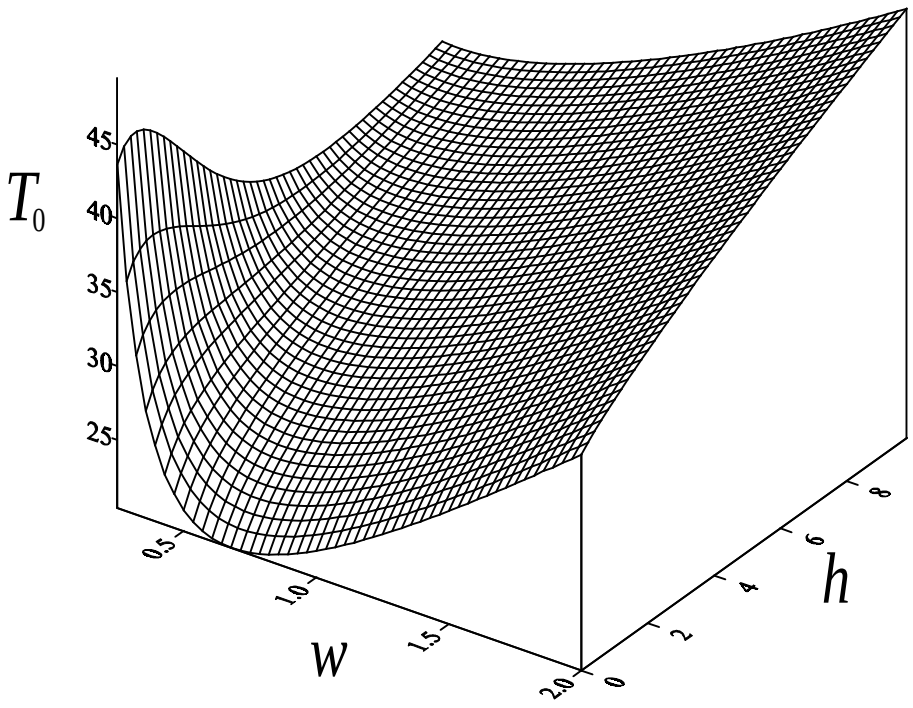

b
χ
χ0
χ0
2
χ
1
0.2
0.1
20
Tf
T0
40
T

Fig.3

a


T_0
45
40
35
30
25
0.5
1.0
1.5
2.0
w
0
2
4
6
8
h

b

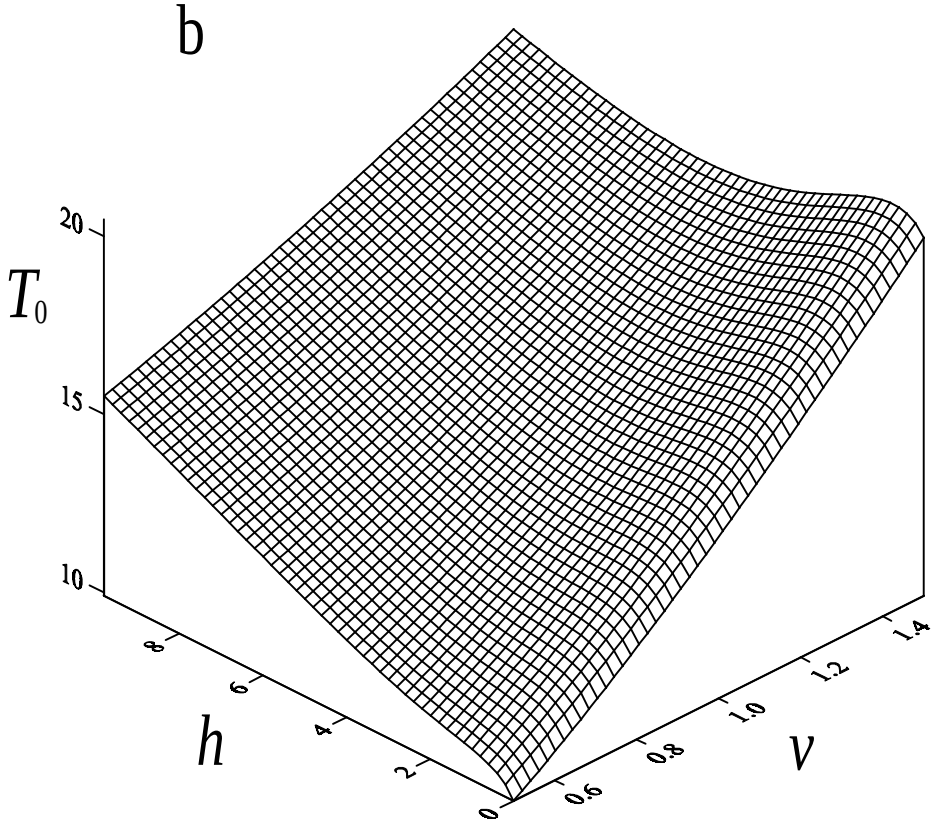

T_0
20
15
10
8
6
4
2
0
h
0.6
0.8
1.0
1.2
1.4
v

Fig.4

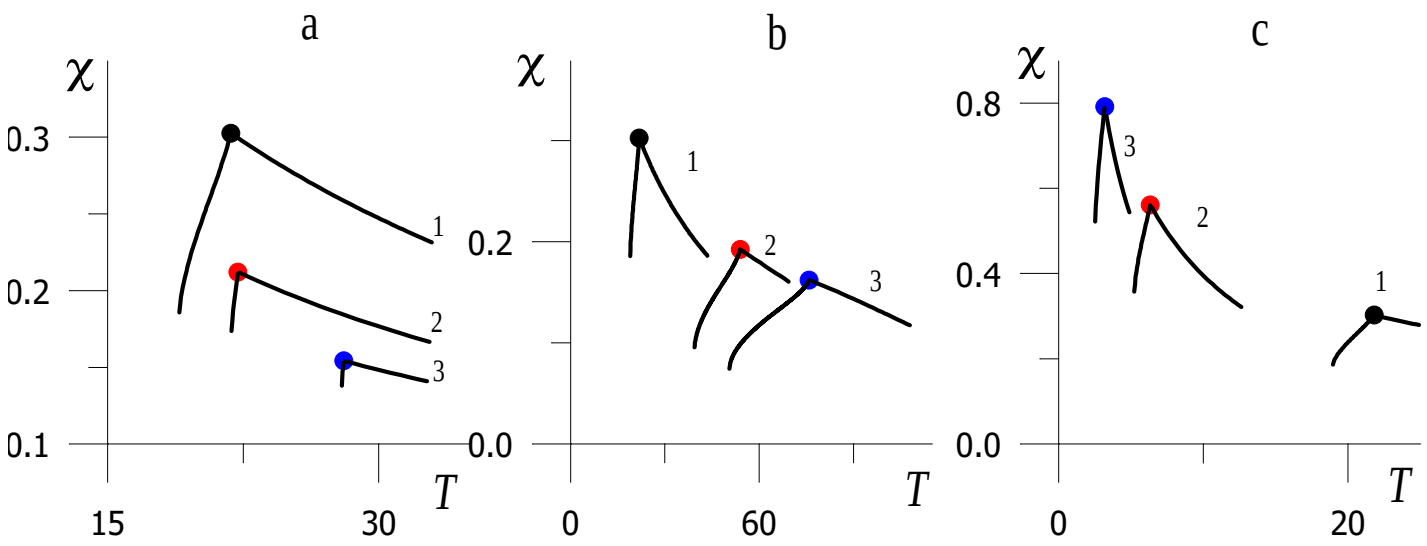
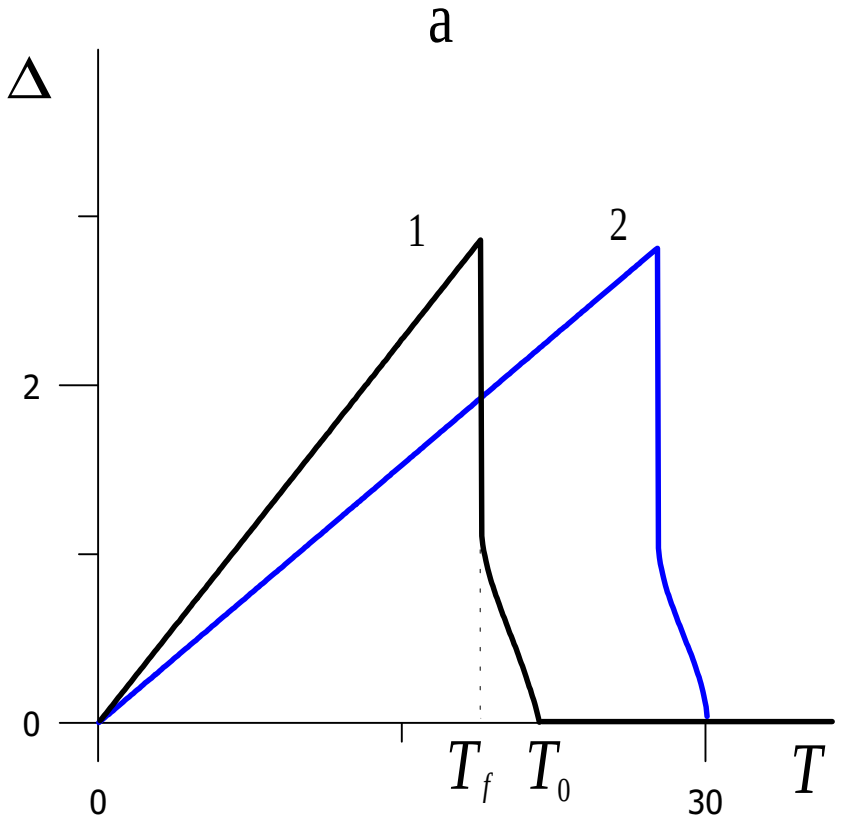


Fig.5



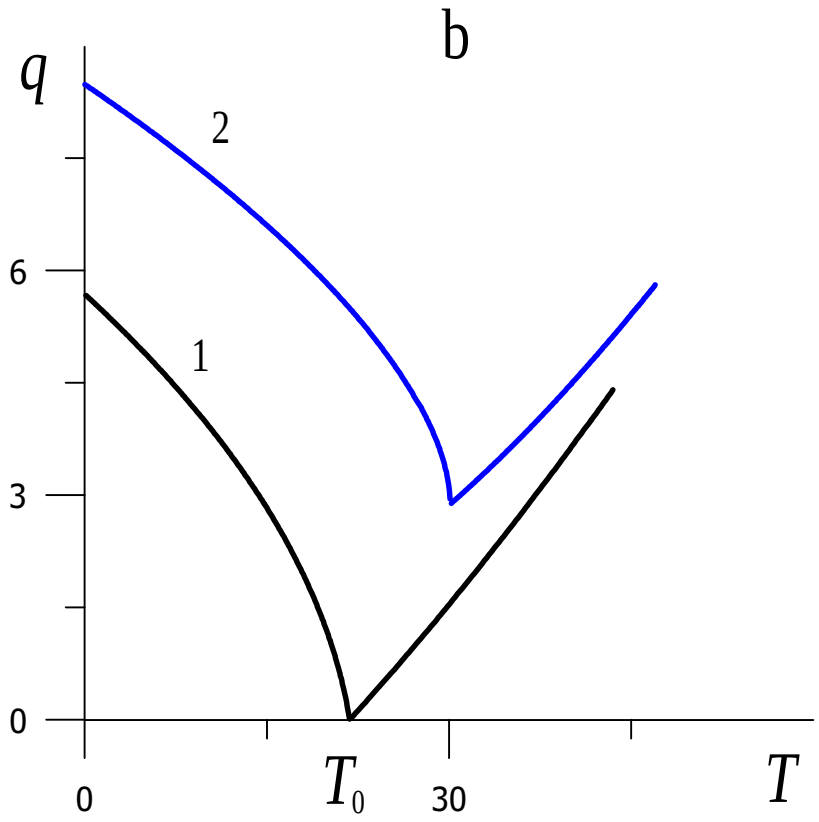


Fig.6